# Convergence of skew-symmetric iterative methods


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Applied Linear Algebra
In honor of Hans Schneider
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$$Au = f, \tag{1}$$

$$B(\omega)\frac{y^{n+1} - y^n}{\tau} + Ay^n = f, \quad n \geq 0, \tag{2}$$

where $f$, $y_0 \in H$,$H$ is $n$-dimensional real Hilbert space, $f$ is the right part of (1), $A$, $B(\omega)$ are linear operators (matrices) in $H$, $A$ is given by equation (1), $B(\omega)$ is invertible, $y_0$ is an initial guess, $y_k$ is the $k$-th approach, $\tau, \omega > 0$ are iterative parameters, $u$ is the solution that has been obtained.

$$y^{n+1} = Gy^n + \tau f,$$

$$G = E - \tau B^{-1}(\omega)A,$$
$$G = E - \tau F, F = B^{-1}(\omega)A. \tag{3}$$

# Skew-symmetric decomposition

$$A = A_0 + A_1,$$
$$A_0 = (A + A^*)/2, \quad (4)$$
$$A_1 = \frac{1}{2}(A - A^*) = -A_1^*,$$

## L.A. Krukier (1979)

$$A_1 = K_L + K_U, K_L = -K_U^*, \quad (5)$$

## Z.Z. Bai, L. Wang (2004)

$$A_1 = K_L + H_0 + K_U - H_0 = \widetilde{K}_L + \widetilde{K}_U,$$
$$\widetilde{K}_L = K_L + H_0, \widetilde{K}_U = K_U - H_0 \quad (6)$$

# Skew-symmetric iterative methods - main idea

$$B(\omega) = B_C + \omega((1+j)K_L + (1-j)K_U), \quad j = \pm 1, \ B_c = B_c^* \tag{7}$$

$$B(\omega) = (B_C + \omega K_U)B_C^{-1}(B_C + \omega K_L), \tag{8}$$

where $B_C = B_C^*$ can be chosen arbitrarily, but has to be symmetric.

$$B = B_0 + B_1, B_0 = B_0^T, B_1 = -B_1^T \tag{9}$$

$$B_0 = B_c + \omega(K_H - K_B) \tag{10}$$

$$B_0 = B_c + \omega^2 K_H B_c^{-1} K_B \tag{11}$$

$$B_1 = \omega A_1 \tag{12}$$

## Definitions

The linear system (1) is called strongly non-symmetric if

$$SSC = \|A_0\| \,/\, \|A_1\| \sim O(1), \tag{13}$$

where $\|*\|$ is one of matrix norms.

### The field of values

$$H(A) = \left\{x^* A x : x \in \mathcal{C}^n, x^* x = 1\right\}$$

$$Sp(A) \subseteq H(A)$$

### Field of values of the matrix $\Psi$ through $R(\Psi)$.

$$\begin{cases} \alpha = \alpha(A_0) = \frac{(A_0 x, x^*)}{(x, x^*)} \in R(A_0) \\ \beta = \beta(B_0) = \frac{(B_0 x, x^*)}{(x, x^*)} \in R(B_0) \\ i\gamma = i\gamma(A_1) = \frac{(A_1 x, x^*)}{(x, x^*)} \in iR(A_1) \end{cases} \tag{14}$$

Let $\lambda_j$ be eigenvalue of a matrix $F = B^{-1}A$ and

$$\lambda_j = Re\lambda_j + iIm\lambda_j.$$

Then

$$|1 - \tau\lambda_j| < 1,$$

$$\forall j, \lambda_j \in Sp(F), F = B^{-1}A, \tau > 0, |\lambda_j|^2 = (Re\lambda_j)^2 + (Im\lambda_j)^2.$$

### Theorem

Let the matrix $F$ be stable. Then exist such value $\tau < \frac{2 \min Re\lambda}{\rho^2}$ for which method (2) converges.

### Corollary

If matrix $A$ is positive real $\left(A_0 = \frac{1}{2}\left(A + A^T\right) > 0\right)$, and matrix $B = B^T > 0$ is SPD then method (2) converges.

### Remark

Unfortunately, it can't be expended on SSIM, since matrix $B$ isn't SPD in our case.

## Generalized eigenvalue problem

$$Fx = \lambda x \Rightarrow B^{-1}Ax = \lambda x$$

$$\Rightarrow Ax = \lambda Bx \qquad (15)$$

$$(A_0 + A_1)x = \lambda(B_0 + \omega A_1)x$$

$$(A_0 - \lambda B_0)x = (\lambda\omega - 1)A_1 x. \qquad (16)$$

# Localizaton of spectrum

$$\frac{(A_0x, x^*)}{(x, x^*)} - \lambda\frac{(B_0x, x^*)}{(x, x^*)} = (\lambda\omega - 1)\frac{(A_1x, x^*)}{(x, x^*)}. \quad (17)$$

$$\begin{cases} R(A_0) \in [\alpha_0, \alpha_1], \alpha_0 = \lambda_{min}(A_0), \alpha_1 = \lambda_{max}(A_0) \\ R(B_0) \in [\beta_0, \beta_1], \beta_0 = \lambda_{min}(B_0), \beta_1 = \lambda_{max}(B_0) \\ R(A_1) \in [-\gamma_1, \gamma_1], \gamma_1 = |\lambda_{max}(A_1)| \end{cases} \quad (18)$$

$$\alpha - \lambda\beta = (\lambda\omega - 1)i\gamma$$

$$\begin{cases} \beta\lambda_0 - \omega\gamma\lambda_1 = \alpha \\ \omega\gamma\lambda_0 + \beta\lambda_1 = \gamma \end{cases} \quad (19)$$

## Connection between spectrum and field of values

$$\begin{cases} \lambda_0 = \frac{\omega\gamma^2+\alpha\beta}{\omega^2\gamma^2+\beta^2} \\ \lambda_1 = \frac{\beta-\omega\gamma}{\omega^2\gamma^2+\beta^2}\gamma \end{cases} \qquad (20)$$

$$D = \omega^2\gamma^2 + \beta^2 \geq 0$$

### Theorem

Let conditions (14) and (18) be satisfied and $\beta_0 > 0$. Then for the real and imaginary parts of eigenvalues $\lambda$ matrices $F = B^{-1}A$ equality (20) is carried out.

$$Ch1 = \omega\gamma^2 + \alpha\beta \tag{21}$$

$$\begin{cases} \alpha \in [\alpha_0, \alpha_1], \\ \beta \in [\beta_0, \beta_1], \beta_0 > 0 \\ \gamma^2 \in [\gamma_0^2, \gamma_1^2], \gamma_0 \geq 0 \end{cases} \tag{22}$$

**if $\alpha_0 > 0$**

$$Ch1 \in [\omega\gamma_0^2 + \alpha_0\beta_0, \omega\gamma_1^2 + \alpha_1\beta_1].$$

So $\lambda_0 > 0$ and the method converges.

**Theorem**

Let matrix $A$ be positive real. Then an iterative method (2) with positive real matrix $B$, for which it is carried out (12), converges.

**Note!**

If $A$ and $B$ are positive real matrices and $B_1 = \omega A_1$ is realized then $AB$ and $BA$ are stable matrices.

## Case: $\alpha_0 < 0$

$$Ch1 \in [\omega\gamma_0^2 - |\alpha_0|\beta_1, \omega\gamma_1^2 + \alpha_1\beta_1]. \tag{23}$$

$$Ch1L = \omega\gamma_0^2 - |\alpha_0|\beta_1 > 0$$

$$\omega > \frac{|\alpha_0|\beta_1}{\gamma_0^2} \tag{24}$$

### Theorem

Let the matrices $A$ and $A_1$ be nonsingulars, the conditions (14) and (18) are carried out. Then if an initial matrix $A$ isn't positive real (in formulas (22) $\alpha_0 < 0$) iterative method (2) with positive real matrix $B$ (in formulas (22) $\beta_0 > 0$), for which the condition (12) is satisfied, converges, if the inequality (24) for parameter $\omega$ is carried out.

THANK YOU
FOR ATTENTION!

СПАСИБО ЗА ВНИМАНИЕ!