Characterization of Alpha1 and Alpha2-matrices

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Outline

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Introduction

Given the matrix $A \in \mathbb{C}^{n\times n}$

- Comparison matrix of A:

$$\mathcal{M}(A) = \begin{cases} -\left|a_{ij}\right|, & \text{if } i \neq j \\ \left|a_{ii}\right|, & \text{if } i = j \end{cases}, \qquad i,j = 1,2,\ldots,n,$$

- Set of equimodular matrices of A:

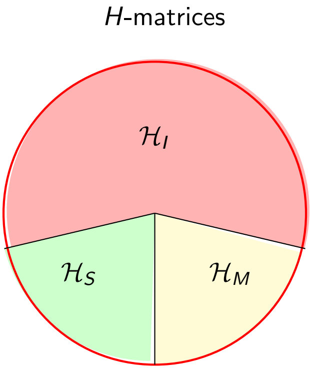
$$\Omega(A) \equiv \{B \in \mathbb{C}^{n\times n} :\ \mathcal{M}(B) = \mathcal{M}(A)\}.$$

- A is an H-matrix if is comparison matrix is an M-matrix, i.e.,

$$\mathcal{M}(A) = sI - C \quad \text{with} \quad C \geq 0 \;\; \text{and} \;\; s \geq \rho(C)$$

Classes of H-matrices

- Invertible class ($\mathcal{H}_I$): A and $\mathcal{M}(A)$ invertible
- Mixed class ($\mathcal{H}_M$): $\mathcal{M}(A)$ singular and in $\Omega(A)$ there is nonsingular matrix
- Singular class ($\mathcal{H}_S$): all matrices in $\Omega(A)$ singular



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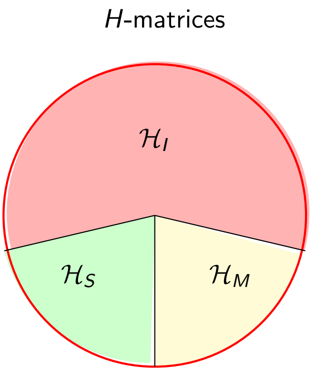

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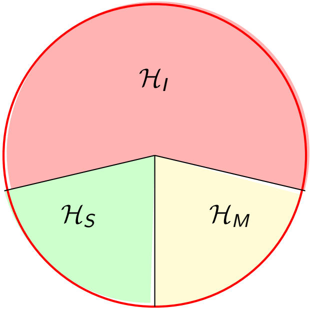
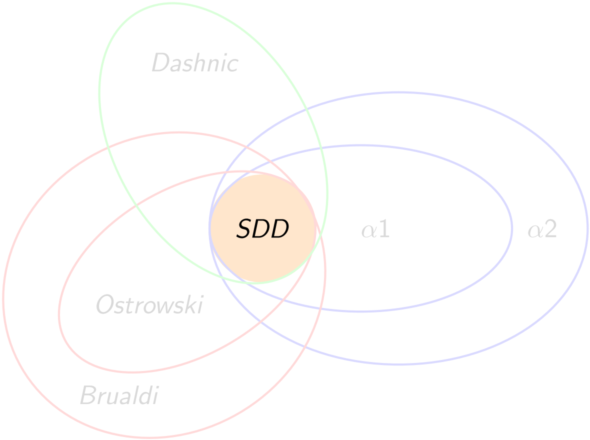
H-matrices



We deal with H-matrices in the invertible class: mainly SUBCLASSES.

Subclasses of invertible H-matrices: motivation

- Identify H-matrices
- Eigenvalue localization
- Application to iterative methods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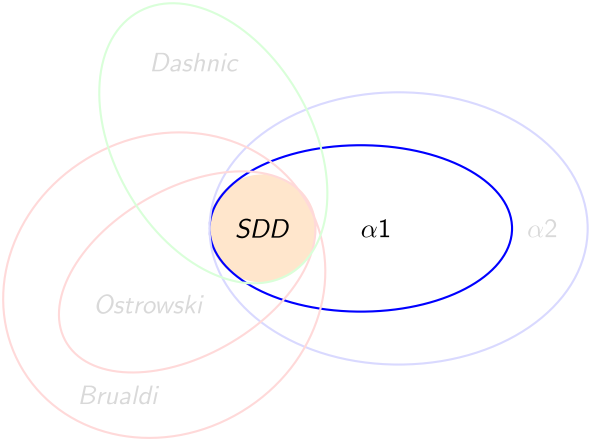
α2
Ostrowski
Brualdi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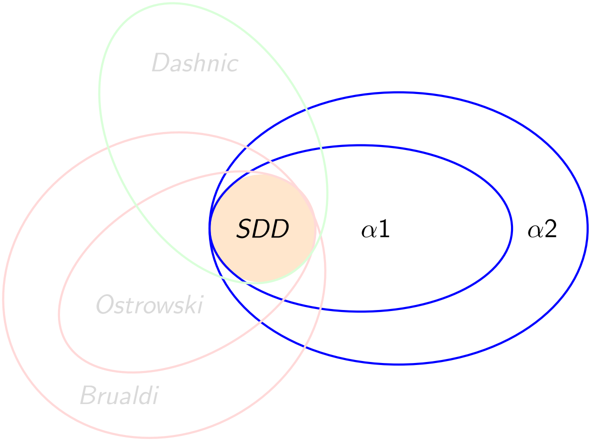
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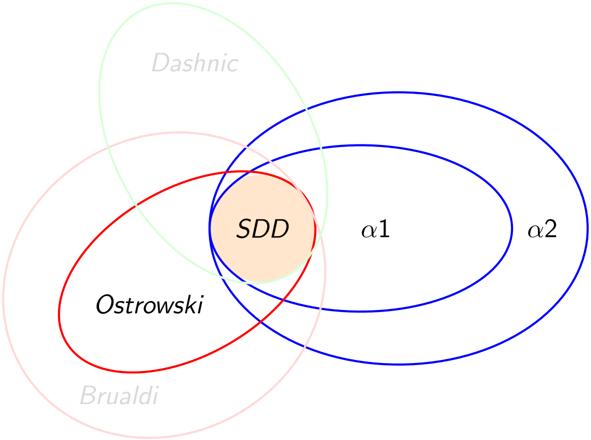
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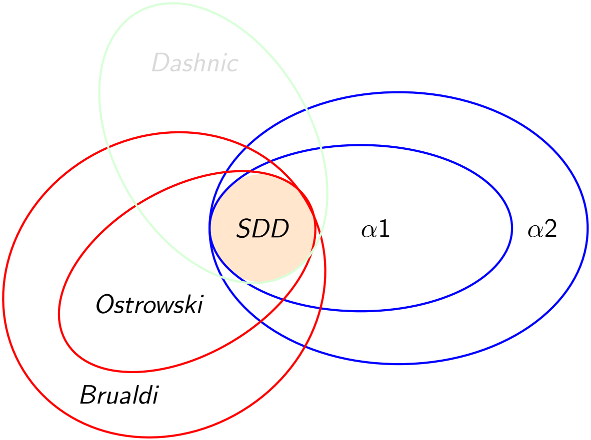
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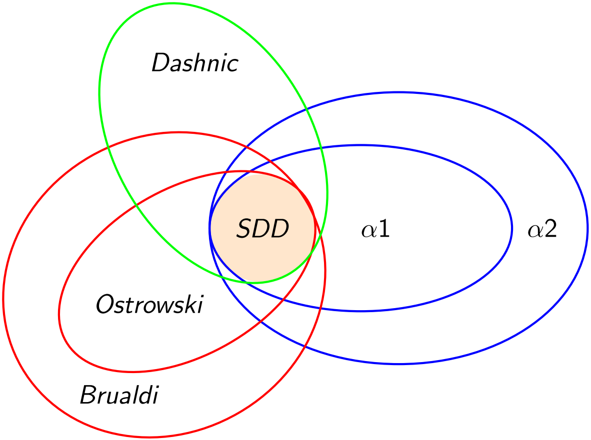
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Some subclasses of invertible H-matrices

Notation

- Set of indices $N = \{1, 2, \ldots, n\}$
- Row sum $r_i := r_i(A) = \sum_{j \in N \setminus i} |a_{ij}|$
- Column sum $c_j := r_j(A^T) = \sum_{i \in N \setminus j} |a_{ij}|$

Subclasses

- SDD $|a_{ii}| > r_i, \quad \forall i \in N$
- $\alpha 1$ $|a_{ii}| > \alpha r_i + (1-\alpha) c_i, \quad \forall i \in N$
- $\alpha 2$ $|a_{ii}| > r_i^{\alpha} c_i^{(1-\alpha)}, \quad \forall i \in N$
- Ostrowski $|a_i||a_j| > r_i r_j, \quad \forall i, j \in N, \quad i \neq j$
- Dashnic $|a_{ii}| \left(|a_{jj} - r_j + |a_{ji}|\right) > r_i |a_{ji}|, \quad j \in N \setminus \{i\}$

Outline

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α1-matrices

Definition

A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, is said to be an α1-matrix if there exists $\alpha \in [0, 1]$, such that

$$|a_{ii}| > \alpha r_i + (1 - \alpha) c_i,$$

for all $i \in N$.

α1-matrices

Definition

Given $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, let us define

$$\begin{aligned} \mathcal{R} &= \{i \in N : \quad r_i > c_i\} \\ \mathcal{C} &= \{i \in N : \quad c_i < r_i\} \\ \mathcal{E} &= \{i \in N : \quad r_i = c_i\}. \end{aligned}$$

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Definition

For any matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, and each $i \in N$, such that $i \notin \mathcal{E}$, we define

$$\phi_i = \frac{|a_{ii}| - c_i}{r_i - c_i} \in \mathbb{R} \qquad (1)$$

α1-matrices

Theorem

Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then A is an α1-matrix if and only if the following conditions hold

(i) $U(A) \cap [0,1] \neq \emptyset$,

(ii) $|a_{ii}| > r_i$, *for all* $i \in \mathcal{C}$.

where $U(A)$ is given by

$$U(A) =]\max_{i\in\mathcal{R}} \phi_i, \min_{i\in\mathcal{C}} \phi_i[$$

Corollary

Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be an α1-matrix, for some parameter α. Then $\alpha \in U(A) \cap [0,1] \neq \emptyset$.

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Theorem

A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, with $n \geq 2$, is an α1-matrix if and only if the following two conditions hold

$$\text{(i)} \quad |a_{ii}| > \min\{r_i, c_i\}, \text{ for all } i \in N,$$

$$\text{(ii)} \quad \frac{|a_{ii}| - c_i}{r_i - c_i} > \frac{c_j - |a_{jj}|}{c_j - r_j}, \text{ for all } i \in \mathcal{R}, \text{ and all } j \in \mathcal{C}.$$

Outline

α2-matrices

Recall the definition of α2-matrices.

Definition

A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, is said to be an α2-matrix if there exists $\alpha \in [0, 1]$, such that

$$|a_{ii}| > r_i^{\alpha} \cdot c_i^{1-\alpha},$$

for all $i \in N$.

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Recall the definition of α2-matrices.

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Definition

For any matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, and each $i \in N$, such that $i \notin \mathcal{E}$, we define

$$\overline{\phi_i} = \frac{\log |a_{ii}| - \log c_i}{\log r_i - \log c_i} \in \mathbb{R} \tag{2}$$

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Theorem

Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then A is an α1-matrix if and only if the following conditions hold

(i) $\overline{U(A)} \cap [0,1] \neq \emptyset$,

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$$\overline{U(A)} =]\max_{i\in\mathcal{R}} \overline{\phi_i}, \min_{i\in\mathcal{C}} \overline{\phi_i}[$$

Corollary

Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be an α1-matrix, for some parameter α. Then $\alpha \in \overline{U(A)} \cap [0,1] \neq \emptyset$.

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Theorem

Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. *Then* A *is an* α*1-matrix if and only if the following conditions hold*

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A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, with $n \geq 2$, is an α1-matrix if and only if the following two conditions hold

(i) $|a_{ii}| > \min\{r_i, c_i\}$, *for all* $i \in N$,

(ii) $$\log_{\frac{r_i}{c_i}} \frac{|a_{ii}|}{c_i} > \log_{\frac{c_j}{r_j}} \frac{c_j}{|a_{jj}|},$$ *for all* $i \in \mathcal{R}$, *for which* $c_i \neq 0$,

and for all $j \in \mathcal{C}$, *for which* $r_j \neq 0$.

Outline

Eigenvalues localization

Theorem

(Geršgorin) Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$*, with* $n \geq 2$*, and let* λ *be an eigenvalue. Then, there exists an index* $i \in N := \{1, 2, ..., n\}$ *such that*

$$|\lambda - a_{ii}| \leq r_i$$

Thus, denoting $\sigma(A)$ *to be the spectrum of the matrix* A*, we have*

$$\sigma(A) \subset \Gamma(A) := \bigcup_{i \in N} \Gamma_i(A),$$

where

$$\Gamma_i(A) := \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i\}.$$

Eigenvalues localization

- Recall $\sigma(A) \subseteq \Gamma(A^T)$, and thus $\sigma(A) \subseteq \Gamma(A) \cap \Gamma(A^T)$
- Define $\overline{\Gamma}_i(A) := \{z \in \mathbb{C} : |z - a_{ii}| \leq \min\{r_i, r_i\}\}$.
- Consider $\overline{\Gamma}(A) := \bigcup_{i \in N} \overline{\Gamma}_i(A)$
- It is clear $\overline{\Gamma}(A) \subseteq \Gamma(A) \cap \Gamma(A^T)$ and $\sigma(A) \notin \overline{\Gamma}(A)$
- What else is needed to $\sigma(A) \subseteq \overline{\Gamma}(A) \bigcup$?

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Eigenvalues localization

Theorem

Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, with $n \geq 2$, and let λ be its eigenvalue. Then, there exists an index $i \in N$ such that $|\lambda - a_{ii}| \leq \min\{r_i, c_i\}$, or, there exist $i \in \mathcal{R}$ and $j \in \mathcal{C}$, such that

$$|\lambda - a_{ii}|\,(c_j - r_j) + |\lambda - a_{jj}|\,(r_i - c_i) \leq c_j r_i - c_i r_j.$$

Thus, we have that

$$\sigma(A) \subset \mathcal{A}_1(A) := \overline{\Gamma}(A) \bigcup \widehat{\Gamma}(A),$$

where $\widehat{\Gamma}(A)$ is given by

$$\widehat{\Gamma}(A) := \bigcup_{\substack{i \in \mathcal{R} \\ j \in \mathcal{C}}} \widehat{\Gamma}_{ij}(A), \quad \text{and}$$

$$\widehat{\Gamma}_{ij}(A) := \{z \in \mathbb{C} : |z - a_{ii}|\,(c_j - r_j) + |z - a_{jj}|\,(r_i - c_i) \leq c_j r_i - c_i r_j\}, \quad (i \in \mathcal{R})\ (j \in \mathcal{C}).$$

Eigenvalues localization

Example

Let

$$A_1 = \begin{pmatrix} 0 & 2 & 0.1 & 0.1 \\ 0 & 0 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0 & 0 \\ 0.1 & 0.1 & 0 & 2 \end{pmatrix}, \textit{ and } A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0.45 & 0 \\ 0 & 0 & i & 1 \\ 0.45 & 0 & 0 & -i \end{pmatrix}.$$

Eigenvalues localization

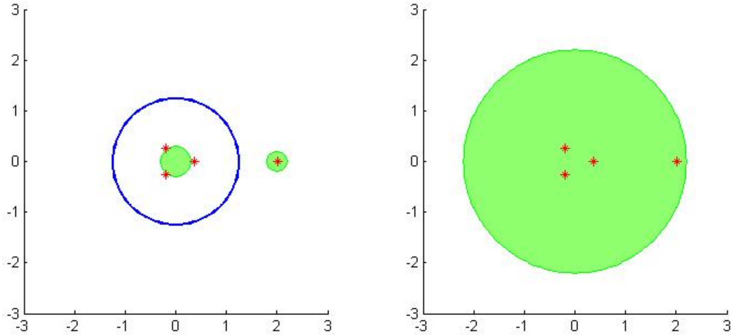
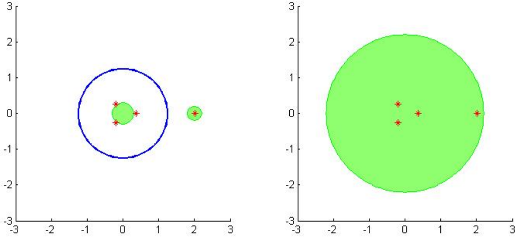


Figure: Inclusion regions for the matrix A_1 of the above example

-Exact eigenvalues are plotted with asterisks.

Left:
- set $\mathcal{A}_1(A_1)$ is shown.
- set $\overline{\Gamma}(A_1)$ is filled.
- set $\widehat{\Gamma}(A_1)$ is the thick boundary.



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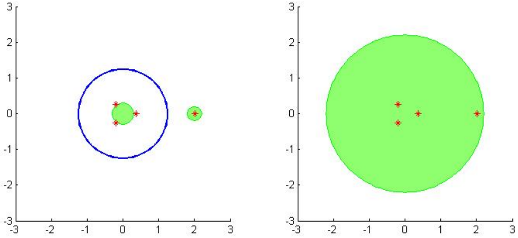
Right:
- intersection of the Geršgorin sets $\Gamma(A_1)$ and $\Gamma(A_1^T)$ is drawn
- $\bigcap$ Geršgorin disks one by one and taking the union fails to capture the eigenvalues.



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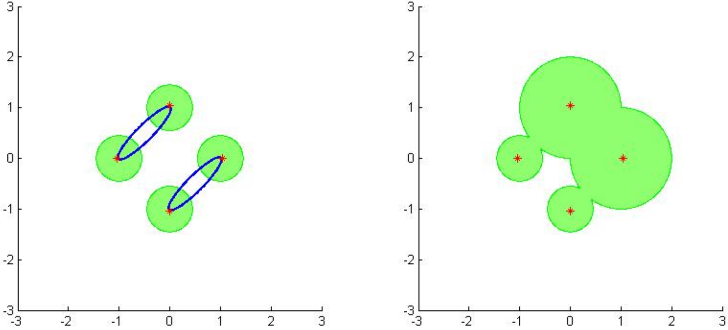
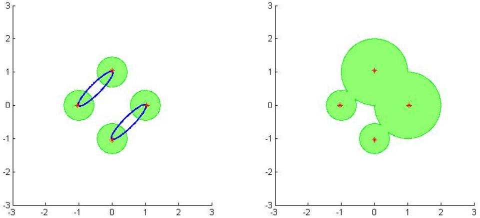


Figure: Inclusion regions for the matrix A_2 of the above example

The Exact eigenvalues are plotted with asterisks.

Left:

- set $\mathcal{A}_1(A_2)$ is shown.
- set $\overline{\Gamma}(A_2)$ is filled.
- set $\widehat{\Gamma}(A_2)$ is the thick boundary.



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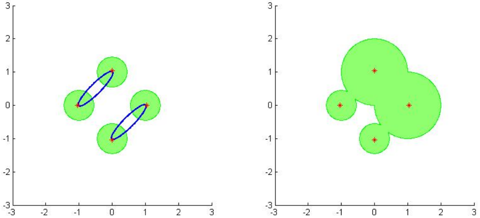

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