Combined matrices of matrices in special classes

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Let A be a nonsingular (complex, or even more general) matrix. In [3], we called the Hadamard (entrywise) product $A \circ (A^{-1})^T$ the combined matrix of A, and denoted it as $C(A)$.

It has good properties:

All its row- and column-sums are equal to one.

If we multiply A by a nonsingular diagonal matrix from the left or from the right, $C(A)$ will not change.

It is a long standing problem to characterize the range of $C(A)$ if A runs through the set of all positive definite $n \times n$ matrices. A partial answer describing the diagonal entries of $C(A)$ in this case was given in [2]:

Theorem 1. *A necessary and sufficient condition that the numbers* a_{11}, a_{22}, $\ldots$, a_{nn} *are diagonal entries of a positive definite matrix and* $\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn}$ *the diagonal entries of its inverse matrix, is fulfilling of :*

1. $a_{ii} > 0$, $\alpha_{ii} > 0$ *for all* i,
2. $a_{ii}\alpha_{ii} - 1 \geq 0$ *for all* i,
3. $2\max_i(\sqrt{a_{ii}\alpha_{ii}} - 1) \leq \sum_i(\sqrt{a_{ii}\alpha_{ii}} - 1)$. In the notation above, it means that a matrix $M = [m_{ik}]$ can serve as $C(A)$ for A positive definite, only if its diagonal entries satisfy $m_{ii} \geq 1$ and $2\max_i(\sqrt{m_{ii}} - 1) \leq \sum_i(\sqrt{m_{ii}} - 1)$.

In fact, there is also another necessary condition, namely, that the matrix $M - I$, I being the identity matrix, is positive semidefinite. Let us show that, for simplicity in the real case. We have for the quadratic form $A \circ A^{-1}$ if $x = (x_1, \ldots, x_n)$, tr meaning the trace and X the *diagonal* matrix $X = \text{diag}(x_i)$,

$$\begin{aligned} \sum_{i,k=1}^{n} a_{ik}\alpha_{ki}x_i x_k &= \sum_{i,k=1}^{n} a_{ik}x_k\alpha_{ki}x_i \\ &= \text{tr}AXA^{-1}X. \end{aligned}$$

Since A is positive definite, it can be written as LL^T, where L is a nonsingular lower triangular matrix. Thus the last expression is $\text{tr}LL^T X(L^T)^{-1}L^{-1}X$, which again is equal to $\text{tr}L^T X(L^T)^{-1}L^{-1}XL$, or $\text{tr}Z^T Z$, where $Z = L^{-1}XL$. The matrix Z is lower triangular and its diagonal is X. Since $\text{tr}Z^T Z$ is the sum of squares of all the entries of Z, it is greater than or equal to $\sum_i x_i^2$. Altogether, $\sum_{i,k=1}^{n} a_{ik}\alpha_{ki}x_i x_k \geq \sum_i x_i^2$.

Let us compare the conditions in Theorem 1 with the fact that $M - I$ is positive semidefinite. We already saw that M has all row- (and column-) sums equal to one, i. e. $Me = e$, where e is the vector of all ones. Since $M - I$ is positive semidefinite, it is Gram matrix of some vectors $u_1, \ldots, u_n$ in a Euclidean space, and by $Me = e$, the sum of these vectors is the zero vector. It follows that they can be considered as vectors forming edges of a closed polygon so that each of the vectors has length not exceeding the sum of lengths of the remaining vectors:

$$|u_i| \leq \sum_{k, k \neq i} |u_k|,$$

i.e. $2|u_i| \leq \sum_k |u_k|$, i.e. $2 \max_i |u_i| \leq \sum_i |u_i|$.
Since $|u_i| = \sqrt{a_{ii}\alpha_{ii} - 1}$, we obtained that

$$2 \max_i \sqrt{a_{ii}\alpha_{ii} - 1} \leq \sum_i \sqrt{a_{ii}\alpha_{ii} - 1}.$$

However, this inequality is weaker than that in $3.$ of Theorem 1.

In [1], a similar problem was considered for the case that A is an $n \times n$ M-matrix, i.e. a real matrix all off-diagonal entries of whose are non-positive but such that for some positive vector u, Au is also positive. The inverse of such matrix exists and is a nonnegative matrix. This implies that $C(A)$ has also all off-diagonal entries non-positive and since $Ae > 0$, the combined matrix of an M-matrix is also an M-matrix. The matrix $C(A) - I$ is then a "singular M-matrix"; it can be obtained as a singular limit of a convergent sequence of M-matrices. We showed that the diagonal entries m_{ii} of $C(A)$ are *characterized* by the conditions
1. $m_{ii} \geq 1$ for all i,
and
2. $2 \max_i (m_{ii} - 1) \leq \sum_i (m_{ii} - 1)$.
Apparently, it is not known whether every M-matrix with row- and column-sums one can serve as combined matrix of some M-matrix A.

Theorem 2. *Let* $A = [a_{ik}]$ *be an arbitrary (even complex) nonsingular tridiagonal matrix,* $A^{-1} = [\alpha_{ik}]$*. Then the sequence* $\{a_{ii}\alpha_{ii} - 1\}$ *has the property that the sum of its odd terms is equal to the sum of the even terms.*

Proof. Let us form the combined matrix $C = A \circ (A^T)^{-1}$. Since all its row and column sums are equal to one, the matrix $C - I$ has all such sums equal to zero. It is also tridiagonal. Let now r_o, r_e, respectively, be the sum of all entries in the odd, respectively even rows of $C - I$, and analogously c_o, c_e the sums of the columns of $C - I$. It is evident that the sum $r_o - r_e + c_o - c_e$, which is zero, is at the same time twice the alternate sum of the diagonal entries of $C - I$.

A symmetric real *Cauchy matrix* A is an $n \times n$ matrix assigned to n real parameters $x_1, \ldots, x_n$, as follows:

$$A = \left[\frac{1}{x_i + x_j} \right], \quad i, j = 1, \ldots, n.$$

It is well known that A is nonsingular if and only if, in addition to the general existence assumption that $x_i + x_j \neq 0$ for all i and j, the x_i's are mutually distinct. In fact, there is a formula for the determinant of A

$$\det A = \frac{\prod_{i,k,i>k} (x_i - x_k)^2}{\prod_{i,j=1}^{n} (x_i + x_j)}. \qquad (1)$$

It is easily seen that such Cauchy matrix is positive definite if all the x_i's are positive. If they are ordered, say $x_1 > x_2 > \cdots > x_n > 0$, then A is *totally positive*, i.e. *all* square submatrices of A have positive determinant. We will again be interested in the combined matrix of C, in particular, in the sequence of its diagonal entries $\{m_{ii}\}$, by proving:

$$\sqrt{m_{11}} - \sqrt{m_{22}} + \sqrt{m_{33}} - + \cdots + (-1)^{n-1}\sqrt{m_{nn}} = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases} \tag{2}$$

We prove first:

Observation *Let $A = [a_{ij}]$ be a nonsingular symmetric Cauchy matrix, $a_{ij} = 1/(x_i + x_j)$. Then there exists a diagonal nonsingular matrix D, such that*

$$A^{-1} = DA^T D. \tag{3}$$

To prove that, let $X = \text{diag}(x_1, \dots, x_n)$. Then, since $x_i a_{ij} + a_{ij} x_j = 1$ for all i, j,

$$XA + AX = J,$$

the matrix of all ones. Multiply this by A^{-1} from both sides. We obtain

$$XA^{-1} + A^{-1}X = A^{-1}ee^T A^{-1}.$$

If we denote the vector $A^{-1}e$ as $(z_1, \dots, z_n)^T$ and the entries of A^{-1} as α_{ij}, the previous equation can be written as

$$x_i \alpha_{ij} + \alpha_{ij} x_j = z_i z_j,$$

i.e. (3):

$$\alpha_{ij} = \frac{z_i z_j}{x_i + x_j}$$

as asserted. Also, the z_i's are all different from zero.

Suppose now that A is totally positive. It is well known that the inverse has then the checkerboard sign pattern, which means that the matrix D will have all positive diagonal entries (or, all negative) if multiplied by the diagonal matrix

$$S = \mathrm{diag}(1, -1, 1, \ldots, (-1)^{n-1}).$$

Let $L = DS$ be this diagonal matrix with positive diagonal. Thus

$$A^{-1} = LSASL,$$

i.e.

$$ALSASL = I,$$

which can be written as

$$L^{\frac{1}{2}} A L^{\frac{1}{2}} S L^{\frac{1}{2}} A L^{\frac{1}{2}} S = I.$$

The matrix $Q = L^{\frac{1}{2}} A L^{\frac{1}{2}}$ has thus the property that

$$(QS)^{-1} = QS,$$

i.e., QS is involutory.

Therefore, the eigenvalues of QS are 1 and -1 only. On the other hand, Q is positive definite, so that the matrix $Q^{\frac{1}{2}}SQ^{\frac{1}{2}}$ which has the same eigenvalues as QS is congruent to S. Thus the eigenvalues of QS are 1 and -1 with the same multiplicity if n is even, the multiplicity of 1 being greater by one if n is odd. Now, if $q_{11}, q_{22}, \ldots, q_{nn}$ are the common diagonal entries of Q and Q^{-1}, then $q_{11}, -q_{22}, \ldots, (-1)^{n-1}q_{nn}$ are the diagonal entries of QS which implies that trace of QS is 0 or 1. Since the combined matrices $C(A)$ and $C(Q)$ are the same, we have, if m_{ii} are their diagonal entries (i.e., $m_{ii} = q_{ii}^2$)

$$\sqrt{m_{11}} - \sqrt{m_{22}} + \sqrt{m_{33}} - + \cdots + (-1)^{n-1}\sqrt{m_{nn}} = \begin{cases} 1 & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}, \end{cases}$$

as we wanted to prove.

This result can be applied to every principal submatrix of the Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & \frac{1}{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n-1} \end{bmatrix},$$

or,

$$H_n = \left[\frac{1}{i+k-1}\right], \quad i, k = 1, \ldots, n.$$

since it is a totally positive Cauchy matrix (choose $x_i = i - \frac{1}{2}$).

Let us show it on an example.

Example. The submatrix

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

of the Hilbert matrix has the inverse

$$\begin{bmatrix} 300 & -900 & 630 \\ -900 & 2880 & -2100 \\ 630 & -2100 & 1575 \end{bmatrix}.$$

Thus

$$A \circ A^{-1} = \begin{bmatrix} 100 & -225 & 126 \\ -225 & 576 & -350 \\ 126 & -350 & 225 \end{bmatrix}$$

is an integral matrix. It clearly satisfies (2).

The involutory matrix QS is then $\mathrm{diag}(\sqrt{30}, \sqrt{120}, \sqrt{105}) A \mathrm{diag}(\sqrt{30}, -\sqrt{120}, \sqrt{105})$, i.e.

$$\begin{bmatrix} 10 & -15 & 3\sqrt{14} \\ 15 & -24 & 5\sqrt{14} \\ 3\sqrt{14} & -5\sqrt{14} & 15 \end{bmatrix}.$$

The trace condition is fulfilled. Observe that the Hadamard power of QS is the modulus of $A \circ A^{-1}$.

With Tom Markham, we tried to characterize the properties of the sequence $\{m_{ii}\}$ for any totally positive matrix A. We observed that for the Hilbert $n \times n$ matrix H_n, this sequence is *unimodal*, i.e. it has a unique peak; the index i for which the maximum is achieved is approximately $n/\sqrt{3} = .577n$. For a general totally positive matrix A, we proved:

Theorem 3. *If A is an $n \times n$ totally positive matrix, $n \geq 3$, then the diagonal entries m_{ii} of the combined matrix $C(A)$ satisfy $m_{11} < m_{22}$ as well as $m_{n-1,n-1} > m_{nn}$.*

Lemma

Let $A = [a_{ik}]$, $i = 1, \ldots, n-1$, $k = 1, \ldots, n$, *be a totally positive matrix. If* $n \geq 3$ *and* $a_{11} = a_{12}$, *then*

$$\det \begin{bmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{23} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n-1,1} & a_{n-1,3} & \cdots & a_{n-1,n} \end{bmatrix} > \det \begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \end{bmatrix} \quad (4)$$

Proof.

We use induction on n. If $n = 3$, the result is true since

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0$$

(and $a_{11} = a_{12}$) implies $a_{22} > a_{21}$ and indeed

$$\det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} > \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}.$$

Suppose now that $n > 3$ and that for $n - 1$ the result holds. Let B be the matrix which coincides with A in all entries except for $a_{n-1,n}$, and such that its entry $\hat{a}_{n-1,n}$ satisfies

$$\det \begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n-1,2} & a_{n-1,3} & \cdots & \hat{a}_{n-1,n} \end{bmatrix} = 0. \qquad (5)$$

For this B, the inequality (4) holds by Theorem **??**, possibly with

Proof.

of Theorem 3. Since multiplication of a row or a column by a positive number does not change the combined matrix, we can assume that $a_{11} = 1$, $a_{12} = 1$, $a_{22} = 1$, and we can also use the adjoint matrix instead of the inverse. Our problem is then to show that in the partitioning of A as

$$A = \begin{bmatrix} 1 & 1 & A_{13} \\ a_{21} & 1 & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

$$\det \begin{bmatrix} 1 & A_{23} \\ A_{32} & A_{33} \end{bmatrix} < \det \begin{bmatrix} 1 & A_{13} \\ A_{31} & A_{33} \end{bmatrix}. \tag{6}$$

By the Lemma, removing the second row from A,

$$\det \begin{bmatrix} 1 & A_{13} \\ A_{32} & A_{33} \end{bmatrix} < \det \begin{bmatrix} 1 & A_{13} \\ A_{31} & A_{33} \end{bmatrix}. \tag{7}$$

Using the Lemma for columns after removing from A the first column, we obtain

Corollary

Let $A = [a_{ik}]$ *be an* $n \times n$ *totally nonnegative nonsingular matrix, let* $A^{-1} = [\alpha_{ik}]$. *Then, if* $n \geq 3$

$$a_{11}\alpha_{11} \leq a_{22}\alpha_{22}, \tag{9}$$

as well as

$$a_{n-1,n-1}\alpha_{n-1,n-1} \geq a_{nn}\alpha_{nn}. \tag{10}$$

There is another nice class of matrices for which the inverse is easily found. In [4], they were called the *complementary basic matrices* (CB-matrices). These are matrices, if of order n, of the form $G_{i_1} G_{i_2} \cdots G_{i_{n-1}}$, where $(i_1, i_2, \ldots, i_{n-1})$ is some permutation of $(1, 2, \ldots, n-1)$ and the matrices G_k, $k = 1, \ldots, n-1$ have the form

$$G_k = \begin{bmatrix} I_{k-1} & & \\ & C_k & \\ & & I_{n-k-1} \end{bmatrix} \qquad (11)$$

for 2×2 matrices C_k.

Let us mention that in [4], the following was proved:

Theorem 4. *The eigenvalues of all the products* $G_{i_1} G_{i_2} \cdots G_{i_{n-1}}$ *coincide.*

We can now add the following:

Theorem 5. *Also, the diagonal entries of all these matrices coincide.*

The following then holds:

Theorem 6. *Let* A *be any of the matrices in Theorem 4. Then the combined matrix of* A *has the property that the products of the diagonal entries in even positions and in odd positions coincide.*

In the previous notation,

$$m_{11} m_{33} \cdots = m_{22} m_{44} \cdots .$$

Problem 1. *Find necessary and sufficient conditions for the sequence of the diagonal entries of* $C(A)$ *if* A *is a totally positive* $n \times n$ *matrix .*

Problem 2. *Characterize the range of* $C(A)$ *if* A *runs through the set of all totally positive* $n \times n$ *matrices.*

References

[1] M. Fiedler, Relations between the diagonal entries of an M-matrix and of its inverse. *Mat.- fyz. časopis SAV* 121962, 123 – 128.

[2] M. Fiedler, Relations between the diagonal elements of two mutually inverse positive definite matrices. *Czechosl. Math. J. 14 (89)*, (1964), 39 – 51.

[3] M. Fiedler, Notes on the Hilbert and Cauchy matrices. *Linear Algebra Appl.* 432(2010), 351 – 356.

[4] M. Fiedler, Complementary basic matrices. *Linear Algebra Appl.* 384(2004), 199 – 206.