Combined matrices of matrices in special classes

Miroslav Fiedler

Institute of Computer Science AS CR

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Let $A$ be a nonsingular (complex, or even more general) matrix. In [3], we called the Hadamard (entrywise) product $A \circ (A^{-1})^T$ the combined matrix of $A$, and denoted it as $C(A)$.

It has good properties:

All its row- and column-sums are equal to one.

If we multiply $A$ by a nonsingular diagonal matrix from the left or from the right, $C(A)$ will not change.
It is a long standing problem to characterize the range of $C(A)$ if $A$ runs through the set of all positive definite $n \times n$ matrices. A partial answer describing the diagonal entries of $C(A)$ in this case was given in [2]:

**Theorem 1.** A necessary and sufficient condition that the numbers $a_{11}, a_{22}, \ldots, a_{nn}$ are diagonal entries of a positive definite matrix and $\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn}$ the diagonal entries of its inverse matrix, is fulfilling of:

1. $a_{ii} > 0$, $\alpha_{ii} > 0$ for all $i$,
2. $a_{ii} \alpha_{ii} - 1 \geq 0$ for all $i$,
3. $2 \max_i (\sqrt{a_{ii} \alpha_{ii}} - 1) \leq \sum_i (\sqrt{a_{ii} \alpha_{ii}} - 1)$. In the notation above, it means that a matrix $M = [m_{ik}]$ can serve as $C(A)$ for $A$ positive definite, only if its diagonal entries satisfy $m_{ii} \geq 1$ and $2 \max_i (\sqrt{m_{ii}} - 1) \leq \sum_i (\sqrt{m_{ii}} - 1)$. 

In fact, there is also another necessary condition, namely, that the matrix $M - I$, $I$ being the identity matrix, is positive semidefinite. Let us show that, for simplicity in the real case. We have for the quadratic form $A \circ A^{-1} \text{ if } x = (x_1, \ldots, x_n)$, $\text{tr}$ meaning the trace and $X$ the diagonal matrix $X = \text{diag}(x_i)$,

$$\sum_{i,k=1}^{n} a_{ik} \alpha_{ki} x_i x_k = \sum_{i,k=1}^{n} a_{ik} x_k \alpha_{ki} x_i = \text{tr}AXA^{-1}X.$$

Since $A$ is positive definite, it can be written as $LL^T$, where $L$ is a nonsingular lower triangular matrix. Thus the last expression is $\text{tr}LL^T X(L^T)^{-1} L^{-1} X$, which again is equal to $\text{tr}L^T X(L^T)^{-1} L^{-1} X L$, or $\text{tr}Z^T Z$, where $Z = L^{-1}X L$. The matrix $Z$ is lower triangular and its diagonal is $X$. Since $\text{tr}Z^T Z$ is the sum of squares of all the entries of $Z$, it is greater than or equal to $\sum_i x_i^2$. Altogether, $\sum_{i,k=1}^{n} a_{ik} \alpha_{ki} x_i x_k \geq \sum_i x_i^2$. 
Let us compare the conditions in Theorem 1 with the fact that $M - I$ is positive semidefinite. We already saw that $M$ has all row- (and column-) sums equal to one, i.e. $Me = e$, where $e$ is the vector of all ones. Since $M - I$ is positive semidefinite, it is Gram matrix of some vectors $u_1, \ldots, u_n$ in a Euclidean space, and by $Me = e$, the sum of these vectors is the zero vector. It follows that they can be considered as vectors forming edges of a closed polygon so that each of the vectors has length not exceeding the sum of lengths of the remaining vectors:

$$|u_i| \leq \sum_{k,k \neq i} |u_k|,$$

i.e. $2|u_i| \leq \sum_k |u_k|$, i.e. $2 \max_i |u_i| \leq \sum_i |u_i|$. Since $|u_i| = \sqrt{a_{ii} \alpha_{ii} - 1}$, we obtained that

$$2 \max_i \sqrt{a_{ii} \alpha_{ii} - 1} \leq \sum_i \sqrt{a_{ii} \alpha_{ii} - 1}.$$

However, this inequality is weaker than that in 3. of Theorem 1.
In [1], a similar problem was considered for the case that \( A \) is an \( n \times n \) \( M \)-matrix, i.e. a real matrix all off-diagonal entries of whose are non-positive but such that for some positive vector \( u \), \( Au \) is also positive. The inverse of such matrix exists and is a nonnegative matrix. This implies that \( C(A) \) has also all off-diagonal entries non-positive and since \( Ae > 0 \), the combined matrix of an \( M \)-matrix is also an \( M \)-matrix. The matrix \( C(A) - I \) is then a "singular \( M \)-matrix"; it can be obtained as a singular limit of a convergent sequence of \( M \)-matrices. We showed that the diagonal entries \( m_{ii} \) of \( C(A) \) are characterized by the conditions

1. \( m_{ii} \geq 1 \) for all \( i \),

and

2. \( 2 \max_{i}(m_{ii} - 1) \leq \sum_{i}(m_{ii} - 1) \).

Apparently, it is not known whether every \( M \)-matrix with row- and column-sums one can serve as combined matrix of some \( M \)-matrix \( A \).
Theorem 2. Let $A = [a_{ik}]$ be an arbitrary (even complex) nonsingular tridiagonal matrix, $A^{-1} = [\alpha_{ik}]$. Then the sequence $\{a_{ii}\alpha_{ii} - 1\}$ has the property that the sum of its odd terms is equal to the sum of the even terms.

Proof. Let us form the combined matrix $C = A \circ (A^T)^{-1}$. Since all its row and column sums are equal to one, the matrix $C - I$ has all such sums equal to zero. It is also tridiagonal. Let now $r_o, r_e$, respectively, be the sum of all entries in the odd, respectively even rows of $C - I$, and analogously $c_o, c_e$ the sums of the columns of $C - I$. It is evident that the sum $r_o - r_e + c_o - c_e$, which is zero, is at the same time twice the alternate sum of the diagonal entries of $C - I$. 
A symmetric real Cauchy matrix $A$ is an $n \times n$ matrix assigned to $n$ real parameters $x_1, \ldots, x_n$, as follows:

$$A = \begin{bmatrix} \frac{1}{x_i + x_j} \end{bmatrix}, \quad i, j = 1, \ldots, n.$$ 

It is well known that $A$ is nonsingular if and only if, in addition to the general existence assumption that $x_i + x_j \neq 0$ for all $i$ and $j$, the $x_i$’s are mutually distinct. In fact, there is a formula for the determinant of $A$

$$\det A = \frac{\prod_{i,k,i>k}(x_i - x_k)^2}{\prod_{i,j=1}^n (x_i + x_j)}. \quad (1)$$
It is easily seen that such Cauchy matrix is positive definite if all the $x_i$’s are positive. If they are ordered, say $x_1 > x_2 > \cdots > x_n > 0$, then $A$ is totally positive, i.e. all square submatrices of $A$ have positive determinant. We will again be interested in the combined matrix of $C$, in particular, in the sequence of its diagonal entries $\{m_{ii}\}$, by proving:

$$\sqrt{m_{11}}-\sqrt{m_{22}}+\sqrt{m_{33}}-\cdots+(-1)^{n-1}\sqrt{m_{nn}} = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

(2)

We prove first:

**Observation** Let $A = [a_{ij}]$ be a nonsingular symmetric Cauchy matrix, $a_{ij} = 1/(x_i + x_j)$. Then there exists a diagonal nonsingular matrix $D$, such that

$$A^{-1} = D A^T D.$$  

(3)
To prove that, let $X = \text{diag}(x_1, \ldots, x_n)$. Then, since $x_ia_{ij} + a_{ij}x_j = 1$ for all $i, j$,

$$XA + AX = J,$$

the matrix of all ones. Multiply this by $A^{-1}$ from both sides. We obtain

$$XA^{-1} + A^{-1}X = A^{-1}ee^TA^{-1}.$$

If we denote the vector $A^{-1}e$ as $(z_1, \ldots, z_n)^T$ and the entries of $A^{-1}$ as $\alpha_{ij}$, the previous equation can be written as

$$x_i\alpha_{ij} + \alpha_{ij}x_j = z_i z_j,$$

i.e. (3):

$$\alpha_{ij} = \frac{z_i z_j}{x_i + x_j}$$

as asserted. Also, the $z_i$’s are all different from zero.
Suppose now that $A$ is totally positive. It is well known that the inverse has then the checkerboard sign pattern, which means that the matrix $D$ will have all positive diagonal entries (or, all negative) if multiplied by the diagonal matrix
\[ S = \text{diag}(1, -1, 1, \ldots, (-1)^{n-1}). \]

Let $L = DS$ be this diagonal matrix with positive diagonal. Thus
\[ A^{-1} = LSASL, \]
i.e.
\[ ALSASL = I, \]
which can be written as
\[ L^{\frac{1}{2}} AL^{\frac{1}{2}} SL^{\frac{1}{2}} AL^{\frac{1}{2}} S = I. \]

The matrix $Q = L^{\frac{1}{2}} AL^{\frac{1}{2}}$ has thus the property that
\[ (QS)^{-1} = QS, \]
i.e., $QS$ is involutory.
Therefore, the eigenvalues of $QS$ are 1 and $-1$ only. On the other hand, $Q$ is positive definite, so that the matrix $Q^{\frac{1}{2}} SQ^{\frac{1}{2}}$ which has the same eigenvalues as $QS$ is congruent to $S$. Thus the eigenvalues of $QS$ are 1 and $-1$ with the same multiplicity if $n$ is even, the multiplicity of 1 being greater by one if $n$ is odd. Now, if $q_{11}, q_{22}, \ldots, q_{nn}$ are the common diagonal entries of $Q$ and $Q^{-1}$, then $q_{11}, -q_{22}, \ldots, (-1)^{n-1} q_{nn}$ are the diagonal entries of $QS$ which implies that trace of $QS$ is 0 or 1. Since the combined matrices $C(A)$ and $C(Q)$ are the same, we have, if $m_{ii}$ are their diagonal entries (i.e., $m_{ii} = q_{ii}^2$)

$$\sqrt{m_{11}} - \sqrt{m_{22}} + \sqrt{m_{33}} - \cdots + (-1)^{n-1} \sqrt{m_{nn}} = \begin{cases} 1 & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}, \end{cases}$$

as we wanted to prove.
This result can be applied to every principal submatrix of the Hilbert matrix

\[
\begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots & \frac{1}{n} \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \ldots & \frac{1}{n+1} \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \ldots & \frac{1}{n+2} \\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \ldots & \ldots \\
\frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \ldots & \ldots \\
\frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \ldots & \ldots \\
\frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \ldots & \frac{1}{2n-1}
\end{bmatrix},
\]

or,

\[
H_n = \begin{bmatrix}
\frac{1}{i + k - 1}
\end{bmatrix}, \quad i, k = 1, \ldots, n.
\]

since it is a totally positive Cauchy matrix (choose \( x_i = i - \frac{1}{2} \)).
Let us show it on an example.

**Example.** The submatrix

\[
A = \begin{bmatrix}
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\
\frac{1}{5} & \frac{1}{6} & \frac{1}{7}
\end{bmatrix}
\]

of the Hilbert matrix has the inverse

\[
\begin{bmatrix}
300 & -900 & 630 \\
-900 & 2880 & -2100 \\
630 & -2100 & 1575
\end{bmatrix}.
\]

Thus

\[
A \circ A^{-1} = \begin{bmatrix}
100 & -225 & 126 \\
-225 & 576 & -350 \\
126 & -350 & 225
\end{bmatrix}
\]

is an integral matrix. It clearly satisfies (2).
The involutory matrix $QS$ is then $\text{diag}(\sqrt{30}, \sqrt{120}, \sqrt{105})A\text{diag}(\sqrt{30}, -\sqrt{120}, \sqrt{105})$, i.e.

\[
\begin{bmatrix}
10 & -15 & 3\sqrt{14} \\
15 & -24 & 5\sqrt{14} \\
3\sqrt{14} & -5\sqrt{14} & 15
\end{bmatrix}.
\]

The trace condition is fulfilled. Observe that the Hadamard power of $QS$ is the modulus of $A \circ A^{-1}$. 
With Tom Markham, we tried to characterize the properties of the sequence \( \{m_{ii}\} \) for any totally positive matrix \( A \). We observed that for the Hilbert \( n \times n \) matrix \( H_n \), this sequence is \textit{unimodal}, i.e. it has a unique peak; the index \( i \) for which the maximum is achieved is approximately \( n/\sqrt{3} = 0.577n \). For a general totally positive matrix \( A \), we proved:

**Theorem 3.** If \( A \) is an \( n \times n \) totally positive matrix, \( n \geq 3 \), then the diagonal entries \( m_{ii} \) of the combined matrix \( C(A) \) satisfy \( m_{11} < m_{22} \) as well as \( m_{n-1,n-1} > m_{nn} \).
Lemma

Let $A = [a_{ik}], i = 1, \ldots, n - 1, k = 1, \ldots, n,$ be a totally positive matrix. If $n \geq 3$ and $a_{11} = a_{12}$, then

$$
\begin{vmatrix}
    a_{11} & a_{13} & \cdots & a_{1n} \\
    a_{21} & a_{23} & \cdots & a_{2n} \\
    \cdot & \cdot & \cdots & \cdot \\
    a_{n-1,1} & a_{n-1,3} & \cdots & a_{n-1,n}
\end{vmatrix} >

\begin{vmatrix}
    a_{12} & a_{13} & \cdots & a_{1n} \\
    a_{22} & a_{23} & \cdots & a_{2n} \\
    \cdot & \cdot & \cdots & \cdot \\
    a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n}
\end{vmatrix}

(4)
Proof.
We use induction on $n$. If $n = 3$, the result is true since
\[
\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0
\]
(and $a_{11} = a_{12}$) implies $a_{22} > a_{21}$ and indeed
\[
\det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} > \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}.
\]

Suppose now that $n > 3$ and that for $n - 1$ the result holds. Let $B$ be the matrix which coincides with $A$ in all entries except for $a_{n-1,n}$, and such that its entry $\hat{a}_{n-1,n}$ satisfies
\[
\det \begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-1,2} & a_{n-1,3} & \cdots & \hat{a}_{n-1,n} \end{bmatrix} = 0. \tag{5}
\]

For this $B$, the inequality (4) holds by Theorem ??, possibly with
Proof.

of Theorem 3. Since multiplication of a row or a column by a positive number does not change the combined matrix, we can assume that $a_{11} = 1$, $a_{12} = 1$, $a_{22} = 1$, and we can also use the adjoint matrix instead of the inverse. Our problem is then to show that in the partitioning of $A$ as

$$A = \begin{bmatrix} 1 & 1 & A_{13} \\ a_{21} & 1 & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

$$\det \begin{bmatrix} 1 & A_{13} \\ A_{23} & A_{33} \end{bmatrix} < \det \begin{bmatrix} 1 & A_{13} \\ A_{31} & A_{33} \end{bmatrix}. \quad (6)$$

By the Lemma, removing the second row from $A$,

$$\det \begin{bmatrix} 1 & A_{13} \\ A_{31} & A_{33} \end{bmatrix} < \det \begin{bmatrix} 1 & A_{13} \\ A_{32} & A_{33} \end{bmatrix}. \quad (7)$$

Using the Lemma for columns after removing from $A$ the first column, we obtain

$$\det \begin{bmatrix} 1 & A_{13} \\ A_{31} & A_{33} \end{bmatrix} < \det \begin{bmatrix} 1 & A_{13} \\ A_{32} & A_{33} \end{bmatrix}. \quad (8)$$

From (6) and (8), the required inequality (6), and thus (9), follows.

The inequality (10) follows from the fact that the transformation $A \rightarrow JAJ$, where $J$ is the skew identity, does not change the property of $A$ to be totally positive. The rest is obvious.
Corollary

Let $A = [a_{ik}]$ be an $n \times n$ totally nonnegative nonsingular matrix, let $A^{-1} = [\alpha_{ik}]$. Then, if $n \geq 3$

$$a_{11} \alpha_{11} \leq a_{22} \alpha_{22},$$

(9)

as well as

$$a_{n-1,n-1} \alpha_{n-1,n-1} \geq a_{nn} \alpha_{nn}.$$  

(10)
There is another nice class of matrices for which the inverse is easily found. In [4], they were called the *complementary basic matrices* (CB-matrices). These are matrices, if of order \( n \), of the form \( G_{i_1} G_{i_2} \cdots G_{i_{n-1}} \), where \( (i_1, i_2, \ldots, i_{n-1}) \) is some permutation of \( (1, 2, \ldots, n - 1) \) and the matrices \( G_k \), \( k = 1, \ldots, n - 1 \) have the form

\[
G_k = \begin{bmatrix} I_{k-1} & C_k \\ C_k & I_{n-k-1} \end{bmatrix}
\]  

(11)

for \( 2 \times 2 \) matrices \( C_k \).
Let us mention that in [4], the following was proved:

**Theorem 4.** The eigenvalues of all the products $G_{i_1} G_{i_2} \cdots G_{i_{n-1}}$ coincide.

We can now add the following:

**Theorem 5.** Also, the diagonal entries of all these matrices coincide.

The following then holds:

**Theorem 6.** Let $A$ be any of the matrices in Theorem 4. Then the combined matrix of $A$ has the property that the products of the diagonal entries in even positions and in odd positions coincide. In the previous notation,

$$m_{11} m_{33} \cdots = m_{22} m_{44} \cdots.$$
**Problem 1.** Find necessary and sufficient conditions for the sequence of the diagonal entries of $C(A)$ if $A$ is a totally positive $n \times n$ matrix.

**Problem 2.** Characterize the range of $C(A)$ if $A$ runs through the set of all totally positive $n \times n$ matrices.

**References**


