Using Matrix Pencils to Solve Discrete Sturm-Liouville Problems with Nonlinear Boundary Conditions

Michael K Wilson
Aihua Li

Montclair State University
New Jersey, USA

ALA, Novi Sad

May 27, 2010

Outline

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- Continuous SLBVP Versus Discrete SLBVP
- Statement of the Problem - DSLBVP with Cubic Boundary Conditions
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- Reducing DSLBVP to an Eigenvalue Problem of Matrix Pencils
- Reducing DSLBVP to Regular Eigenvalue Problems
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Sturm-Liouville Equation

- Named after Jacques Charles Franois Sturm (1803 – 1855) and Joseph Liouville (1809 – 1882).
- Second-order linear differential equation of the form

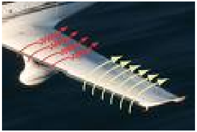
$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=\lambda w(x)y.$$

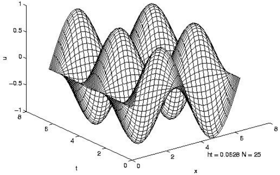
- λ is a parameter.
- $p(x)$, $q(x)$ are positive real functions; $w(x)$ is called a weight function (real).
- A solution pair (λ, y) is called an eigen pair; λ: eigenvalue; y: eigenfunction corresponding to λ.

Applications of Sturm-Liouville Problems

Applications have appeared in science and engineering.

- Belinskiy and Dauer studied SLBVPs that arises in the study of waves of ice-covered oceans.
- Greenburg and Babuska investigated numerical solutions to SLBVPs that arise in acoustical problems.
- Belinskiy and Graef considered a nonlinear SLBVP that arises from the study of the torsion of a wing in a flow.
- Freiling and Yurko discussed inverse problems for a wave equation with a focused source of disturbance.





Converting the problem into Discrete Form

- Continuous Sturm-Liouville Boundary Problem on $[0, b]$

$$\begin{cases} Ly = (1/r)(-py')' + qy = \lambda y \\ y(0) = 0, \\ C(\lambda)y(b) = D(\lambda)y'(b)). \end{cases}$$

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- Let $T = [0, t_1, \dots, t_{N-1}, 1]$. Denote y_n for $y(t_n)$ and $h = t_{n+1} - t_n$. Define

$$\Delta y_n = \frac{y_{n+1} - y_n}{h} \quad \text{and} \quad \nabla y_n = \frac{y_n - y_{n-1}}{h}$$

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- The DSLBVP is of the following form, where L is the operator:

$$\begin{cases} Ly_n = \frac{1}{r_n}\left(\nabla(-p_n \Delta y_n) + q_n y_n\right) = \lambda y_n \\ y_0 = 0 \\ C(\lambda)y_N = D(\lambda)(-p_{N-1}\Delta y_{N-1}). \end{cases}$$

We simplify the operator as

$$Ly_n = -ay_{n+1} + \sigma y_n - ay_{n-1},$$

where $p_n = p, q_n = q, r_n = r$ are constants and $a = -\frac{p}{rh^2}$, and $\sigma = 2\frac{p}{rh^2} + q$.

When $C(\lambda)$ and $D(\lambda)$ are both linear in λ, we rewrite the boundary condition as $-(b_1 y_N + b_2 y_{N-1}) = \lambda(a_1 y_N + a_2 y_{N-1})$ and define

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}.$$

Theorem (Linease case - Harmsen and Li, 2003)

If M is positive definite. Then:

1. *All eigenvalues of the DSLBVP with linear BVC are real and simple.*
2. *The eigenvectors can be chosen to be real.*
3. *An Expansion Theory is given.*

The Quadratic Case

The boundary condition is

$$\left(c_0 + c_1\lambda + c_2\lambda^2\right) y_N = \left(d_0 + d_1\lambda + d_2\lambda^2\right)\left(-p_{N-1}\Delta y_{N-1}\right).$$

We then define

$$M = \begin{pmatrix} h & l \\ l & m \end{pmatrix},$$

where

$$-l = \begin{vmatrix} d_2 & d_0 \\ c_2 & c_0 \end{vmatrix}, \qquad m = \begin{vmatrix} d_2 & d_1 \\ c_2 & c_1 \end{vmatrix}, \qquad h = \begin{vmatrix} d_1 & d_0 \\ c_1 & c_0 \end{vmatrix}.$$

Theorem (Quadratic case - Harmsen and Li, 2007)

If M is positive definite. Then:

1. *All eigenvalues of the DSLBVP with linear BVC are real and simple.*
2. *The eigenvectors can be chosen to be real.*
3. *An Expansion Theory is given.*

Cubic Case: Statement

We focus on DSLBVP with cubic nonlinearity in the BVC:

$$(1)\begin{cases} Ly_n = -ay_{n+1} + \sigma y_n + -ay_{n-1} = \lambda y \\ y(0) = 0 \\ C(\lambda)y_N = D(\lambda)(-p\Delta y_{N-1}). \end{cases}$$

We set:
$C(\lambda) = c_o + c_1\lambda + c_2\lambda^2 + c_3\lambda^3$ and $D(\lambda) = d_o + d_1\lambda + d_2\lambda^2 + d_3\lambda^3$.

Define:

$$\begin{aligned} \alpha(\lambda) = \frac{p}{h}D(\lambda) &= \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 \\ \beta(\lambda) = C(\lambda) + \frac{p}{h}D(\lambda) &= \beta_3\lambda^3 + \beta_2\lambda^2 + \beta_1\lambda + \beta_0. \end{aligned}$$

In particular, $\alpha_0 = \frac{p}{h}d_0$ and $\beta_0 = c_0 + \alpha_0$.

Matrix Form of DSLBVP

- The matrix equation for the DSLBVP is:

$$\Gamma_\lambda \mathbf{y} = \mathbf{0} \text{ where } \mathbf{y}^T = (y_1, y_2, \ldots, y_N).$$

$$\Gamma_\lambda = \begin{bmatrix} \sigma-\lambda & -a & 0 & \cdots & \cdots & 0 \\ -a & \sigma-\lambda & -a & \cdots & \cdots & 0 \\ 0 & -a & \sigma-\lambda & -a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -a & \sigma-\lambda & -a \\ 0 & 0 & \cdots & 0 & \alpha(\lambda) & \beta(\lambda) \end{bmatrix}$$

- DSLBVP has nontrivial solutions if and only if

$$|\Gamma_\lambda| = 0.$$

Cubic Example

Example

For $N = 4$, $(p, q, r) = (1, 0, 16)$ and b.v.c.
$(-1 + 3\lambda + 2\lambda^3)y_4 = (-1 + 2\lambda^3)(-\Delta y_3)$. Thus
$Ly_n = -y_{n+1} + 2y_n - y_{n-1} = \lambda y_n$; we obtain:

$$
\begin{aligned}
Ly_1 &= -y_2 + 2y_1 = \lambda y_1 \\
Ly_2 &= -y_3 + 2y_2 - y_1 = \lambda y_2 \\
Ly_3 &= -y_4 + 2y_3 - y_2 = \lambda y_3 \\
(-1 + 3\lambda + 2\lambda^3)y_4 &= (-1 + 2\lambda^3)(-2(y_4 - y_3)).
\end{aligned}
$$

Example

- From the equations, we obtain the matrix form:

$$\Gamma_\lambda \mathbf{y} = \begin{bmatrix} 2-\lambda & -1 & 0 & 0 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ 0 & 0 & 4-8\lambda^3 & -5+3\lambda+10\lambda^3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \mathbf{0}.$$

$$|\Gamma_\lambda| = 46\lambda - 56\lambda^2 + 39\lambda^3 - 71\lambda^4 + 52\lambda^5 - 10\lambda^6 - 8.$$

Example 2 Continued

Example

- From the equations, we obtain the matrix form:

$$\Gamma_\lambda \mathbf{y} = \begin{bmatrix} 2-\lambda & -1 & 0 & 0 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ 0 & 0 & 4-8\lambda^3 & -5+3\lambda+10\lambda^3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \mathbf{0}.$$

$$|\Gamma_\lambda| = 46\lambda - 56\lambda^2 + 39\lambda^3 - 71\lambda^4 + 52\lambda^5 - 10\lambda^6 - 8.$$

- Using $\lambda = 3.2731$, a root of the polynomial, we solve the matrix equation to obtain:

$$\begin{aligned} y_1 &= 2.0712 \\ y_2 &= -2.6366 \\ y_3 &= 1.2855 \\ y_4 &= 1. \end{aligned}$$

Γ_λ as Cubic Matrix Polynomial

$$\Gamma_\lambda = A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0, \quad \text{where:}$$

$$A_3 = \begin{bmatrix} \mathbf{0}_{N-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & \alpha_3 & \beta_3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mathbf{0}_{N-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & \alpha_2 & \beta_2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -I_{N-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -1 & 0 \\ \mathbf{0} & \alpha_1 & \beta_1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} \sigma & a & 0 & \cdots & \cdots & 0 \\ a & \sigma & a & \cdots & \cdots & 0 \\ 0 & a & \sigma & a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & \sigma & a \\ 0 & 0 & \cdots & 0 & \alpha_0 & \beta_0 \end{bmatrix}.$$

α_n and β_n are the coefficients of the n^{th} term of $\alpha(\lambda)$ and $\beta(\lambda)$ respectively.

DSLBVP Matrix Pencil Eigenvalue Problem

The matrix equation:

$$\Gamma_\lambda \mathbf{y} = (A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0)\mathbf{y} = \mathbf{0}$$

has a nontrivial solution if and only if

$$|A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0| = 0.$$

Moreover, $\Gamma_\lambda \mathbf{y} = \mathbf{0}$ can be written as a matrix pencil equation:

$$(A - \lambda B)\begin{bmatrix} \lambda^N \mathbf{y} \\ \lambda^{N-1}\mathbf{y} \\ \vdots \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

which has non-trivial solutions if and only if

$$|A - \lambda B| = 0.$$

Matrix Pencil

Definition

The expression $A - \lambda B$, where A and B are $m \times n$ matrices, is called a matrix pencil, or just pencil. Here λ is an indeterminate, not a particular numerical value.

Thus eigenvalues of the DSLBVP are the eigenvalues of the pencil $A - \lambda B$ where

$$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix}.$$

Thus we solve

$$\left(\begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix} - \lambda \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \lambda^2 \mathbf{y} \\ \lambda \mathbf{y} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Matrix Pencil

The construction of A and B comes from the following lemma:

Lemma

The N^{th} degree matrix polynomial equation:

$$(A_N\lambda^N + A_{N-1}\lambda^{N-1} + \cdots + A_0)\mathbf{y} = \mathbf{0},$$

is equivalent to

$$(A - \lambda B)\begin{bmatrix} \lambda^N \mathbf{y} \\ \lambda^{N-1}\mathbf{y} \\ \vdots \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

where

$$A = \begin{bmatrix} A_{N-1} & \cdots & A_1 & A_0 \\ & & I_N & \mathbf{0} \\ & \iddots & & \\ I_N & & \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and } B = \begin{bmatrix} -A_N & \mathbf{0} & \cdots & \mathbf{0} \\ & & & I_N \\ & & \iddots & \\ \mathbf{0} & I_N & & \mathbf{0} \end{bmatrix}.$$

DSLBVP as a REP

Assume A is invertible and λ is not equal to zero, from the matrix pencil we have:

$$
\begin{aligned}
|A - \lambda B| = 0 \quad &\Longleftrightarrow \quad \left|\frac{1}{\lambda} A^{-1}(A - \lambda B)\right| = 0 \\
&\Longleftrightarrow \quad \left|\frac{1}{\lambda} I - A^{-1} B\right| = 0 \\
&\Longleftrightarrow \quad \left|A^{-1} B - \mu I\right| = 0, \text{ where } \mu = \frac{1}{\lambda},
\end{aligned}
$$

which is a regular eigenvalue problem (REP).

Obviously, we need A to be invertible in order to be able to solve the DSLBVP as a REP.

Conditions for A_0 to be Nonsingular

Recall that,

- $$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

Conditions for A_0 to be Nonsingular

Recall that,

- $$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix},$$
- A_0 lies on the minor diagonal of the block matrix A and all other blocks on the minor diagonal of A are I_N.

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- Thus $|A| = \pm |A_0| \implies A$ is invertible if and only if A_0 is invertible.

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- A_0 lies on the minor diagonal of the block matrix A and all other blocks on the minor diagonal of A are I_N.
- Thus $|A| = \pm |A_0| \implies A$ is invertible if and only if A_0 is invertible.
- Recall that A_0 is a tridiagonal matrix of the form:

$$A_0 = \begin{bmatrix} \sigma & a & 0 & \cdots & \cdots & 0 \\ a & \sigma & a & \cdots & \cdots & 0 \\ 0 & a & \sigma & a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & \sigma & a \\ 0 & 0 & \cdots & 0 & \alpha_0 & \beta_0 \end{bmatrix},$$

where α_0 and β_0 are the constant terms of $\alpha(\lambda)$ and $\beta(\lambda)$ respectively.

Conditions for A_0 to be Nonsingular

Lemma (Mikkawy and Karawia)

Given the general tridiagonal matrix:

$$T_n = \begin{bmatrix} \sigma_1 & a_1 & 0 & \cdots & \cdots & 0 \\ b_2 & \sigma_2 & a_2 & \cdots & \cdots & 0 \\ 0 & b_3 & \sigma_3 & a_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & b_{n-1} & \sigma_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & b_n & \sigma_n \end{bmatrix},$$

where $a_1 a_2 \cdots a_{n-1} \neq 0$ *and* $b_2 b_3 \cdots b_{n-1} \neq 0$.

$$|T_i| = \begin{cases} 1 & \text{if} \quad i = 0 \\ \sigma_1 & \text{if} \quad i = 1 \\ \sigma_i T_{i-1} - b_i T_{i-1} T_{i-2} & \text{if} \quad i = 2, \quad 3, \cdots, n. \end{cases}$$

Conditions for A_0 to be Nonsingular

Lemma (1)

Given the $(N \times N)$ tridiagonal matrix A_0 as above. Let U_i be the determinant of the $i \times i$ main diagonal of A_0, that is U_i is of the form

$$U_i = \begin{vmatrix} \sigma & a & 0 & \cdots & \cdots & 0 \\ a & \sigma & a & \cdots & \cdots & 0 \\ 0 & a & \sigma & a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & \sigma & a \\ 0 & 0 & \cdots & 0 & a & \sigma \end{vmatrix} \qquad 0 < i \leq N-1.$$

Let $|A_0| = U_N$, then U_N is given iteratively as

$$\begin{cases} U_0 = 1 \\ U_1 = \sigma \\ U_i = \sigma U_{i-1} - a^2 U_{i-2} \quad \text{for} \quad 1 < i < N \\ U_N = \beta_0 U_{N-1} - a\alpha_0 U_{N-2}. \end{cases}$$

Finding Conditions where A_0 is Nonsingular

From Lemma 1

$$U_i = \sigma U_{i-1} - a^2 U_{i-2}$$

We solve

$x^2 - \sigma x + a^2 = 0$ for the roots s_1 and s_2.

Finding Conditions where A_0 is Nonsingular

From Lemma 1

$$U_i = \sigma U_{i-1} - a^2 U_{i-2}$$

We solve

$x^2 - \sigma x + a^2 = 0$ for the roots s_1 and s_2.

- $s_1 \neq s_2$: $\quad U_i = \left(\frac{-s_1}{s_2 - s_1}\right) s_1^i + \left(\frac{s_2}{s_2 - s_1}\right) s_2^i, \qquad 0 \leq i < N$

Finding Conditions where A_0 is Nonsingular

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- $s_1 \neq s_2$: $\quad U_i = \left(\frac{-s_1}{s_2 - s_1}\right) s_1^i + \left(\frac{s_2}{s_2 - s_1}\right) s_2^i, \qquad 0 \leq i < N$
- $s_1 = s_2$: $\quad U_i = s_1^i + i s_2^i, \qquad 0 \leq i < N$

Conditions for A_0 to be Nonsingular

Theorem

Let $U_N = |A_0|$ and s_1, s_2 be the solutions to the equation $x^2 - \sigma x + a^2 = 0$.

1. *In the case $\sigma^2 > 4a^2$, we have $s_1 > s_2 > 0$,*

$$U_N = \frac{1}{s_1 - s_2}\left[\left(\beta_0 - \frac{\alpha_0}{a}s_2\right)s_1^N - \left(\beta_0 - \frac{\alpha_0}{a}s_1\right)s_2^N\right]. \quad (1)$$

2. *In the case $\sigma^2 = 4a^2$, we have $s_1 = s_2$,*

$$U_N = \beta_0 N\left(\frac{\sigma}{2}\right)^{N-1} - \alpha_0 a\,(N-1)\left(\frac{\sigma}{2}\right)^{N-2}. \quad (2)$$

CONDITIONS FOR A_0 TO BE NONSINGULAR

REMARK

Assuming $\alpha_0 = 0$ and $\beta_0 \neq 0$, we note that the tridiagonal matrix A_0 is nonsingular for both cases where $s_1 = s_2$ and $s_1 \neq s_2$. The reason is that when $s_1 \neq s_2$, $U_N = \frac{\beta_0(s_1^N - s_2^N)}{s_1 - s_2}$ is not equal to zero since $\beta_0 \neq 0$. And in the case where $s_1 = s_2$, $U_N = \beta_0 N \left(\frac{\sigma}{2}\right)^{N-1}$ is not equal to zero since $\beta_0 \neq 0$.

THEOREM

Consider the $(N \times N)$ tridiagonal matrix A_0 as above with α_0 not equal to zero. Let $U_N = |A_0|$ and $(\alpha_0, \beta_0) \neq (0, 0)$. Then U_N is not equal to zero in any of the following cases:

1. $s_1 = s_2$ *or*
2. $s_1 \neq s_2$ *but* $\beta_0 - \frac{\alpha_0}{a} s_1 = 0$ *or*
3. $s_1 \neq s_2$ *but* $\beta_0 - \frac{\alpha_0}{a} s_1 \neq 0$ *and* $\frac{\beta_0}{\alpha_0} < \frac{s_2}{a}$.

Conditions for A_0 to be Nonsingular

$s_1 = s_2$.

Case 1: $\beta_0 = 0$. Obvious.

Case 2: $\beta_0 \neq 0$. $U_N = 0$ implies

$$N = \frac{\alpha_0}{\beta_0 + \alpha_0},$$

A contradiction since N is a positive integer. Therefore, $U_N \neq 0$. □

Conditions for A_0 to be Nonsingular

Proof.

(Sketch)
We have two cases to consider when $s_1 \neq s_2$.

Case 1: $\beta_0 - \frac{\alpha_0}{a} s_1 = 0 \Longrightarrow \beta_0 - \frac{\alpha_0}{a} s_2 \neq 0$. Obvious.

Case 2: $\beta_0 - \frac{\alpha_0}{a} s_1 \neq 0$ and $\frac{\beta_0}{\alpha_0} < \frac{s_2}{a}$

$$U_N = 0 \iff \left(\frac{s_1}{s_2}\right)^N = \frac{\beta_0 - \frac{\alpha_0}{a} s_1}{\beta_0 - \frac{\alpha_0}{a} s_2} = \frac{a\beta_0 - \alpha_0 s_1}{a\beta_0 - \alpha_0 s_2}.$$

Further investigation shows that the given condition implies $s_1 < s_2$. This contradicts the fact that

$$s_1 = \frac{\sigma + \sqrt{\sigma^2 - 4a^2}}{2} > s_2 = \frac{\sigma - \sqrt{\sigma^2 - 4a^2}}{2} > 0.$$

□

The Form of DSLBVP as REP

With the results on the non-singularity of A_0 established, we can proceed to discuss $\left|A^{-1}B - \mu I\right|$. We have

$$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix}.$$

thus

$$A^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \\ A_0^{-1} & -A_0^{-1}A_1 & -A_0^{-1}A_2 \end{bmatrix}$$

$$W = A^{-1}B = \begin{bmatrix} \mathbf{0} & I_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ -A_0^{-1}A_3 & -A_0^{-1}A_2 & -A_0^{-1}A_1 \end{bmatrix}$$

We solve $|W - \mu I| = 0$ to determine the solution of the DSLBVP.

We first show that A_0 is similar to a symmetric matrix.

Lemma (4)

If $a\alpha_0 > 0$, then A_0 is similar to a symmetric matrix.

Proof.

Put A_0 in the form

$$A_0 = \begin{bmatrix} E_1 & E_2 \\ E_3 & \beta_0 \end{bmatrix},$$

where

$$E_{1(N-1)\times(N-1)} - \text{major diagonal sub-matrix of } A_0,$$

$$E_2 = [0 \;\cdots\; 0 \; a]^T, \text{ and } E_3 = [0 \;\cdots\; 0 \; \alpha_0].$$

□

Reality of Eigenvalues

Proof.

Define

$$Q = \begin{bmatrix} I_{N-1} & \mathbf{0} \\ \mathbf{0} & \sqrt{\frac{\alpha_0}{a}} \end{bmatrix},$$

then

$$Q^{-1} A_0 Q = \begin{bmatrix} E_1 & \sqrt{\frac{\alpha_0}{a}} E_2 \\ \sqrt{\frac{a}{\alpha_0}} E_3 & \beta_0 \end{bmatrix} = A_0'.$$

Obviously E_1 is symmetric, and $\left(\sqrt{\frac{\alpha_0}{a}} E_2\right)^T = \left[0 \ \cdots \ 0 \ \sqrt{\alpha_0 a}\right] = \sqrt{\frac{a}{\alpha_0}} E_3$.

Therefore A_0 is similar to the symmetric matrix to which we denote A_0'. □

Reality of Eigenvalues

We show an example here to demonstrate Lemma 4.

Example

Take

$$G_2 = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \text{which implies} \quad E_1 = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix},$$

$$E_2 = [0 \quad 0 \ -4]^T, \quad \text{and} \quad E_3 = [0 \quad 0 \ -2]. \text{ Therefore}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ and } Q^{-1}G_2Q = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -2\sqrt{2} \\ 0 & 0 & -2\sqrt{2} & 3 \end{bmatrix}$$

which is obviously symmetric.

Reality of Eigenvalues

Theorem

If $D(\lambda) \equiv d_0 < 0$, then all the eigenvalues of the DSLBVP are real.

Proof.

The eigenvalues of the DSLBVP are the eigenvalues of the matrix pencil $A - \lambda B$. We know from Linear Algebra that if A and B are symmetric matrices with real entries, then the eigenvalues of the pencil are all real. Given

$$D(\lambda) \equiv d_0 < 0, \text{ then } \alpha(\lambda) \equiv \frac{p_{N-1}}{h} d_0 = \alpha_0 < 0.$$

- h and p are both positive
- $a\alpha_0 > 0$, we can apply Lemma 4 to make A_0 similar to a symmetric matrix
- $D(\lambda) = d_0$ implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
- Thus A_1, A_2, A_3 are all diagonal and symmetric matrices.

□

Proof.

Take our

$$Q = \begin{bmatrix} I_{N-1} & \mathbf{0} \\ \mathbf{0} & \sqrt{\frac{\alpha_0}{a}} \end{bmatrix},$$

and make A similar to a symmetric matrix denoted A' by computing:

$$\begin{bmatrix} Q^{-1} & & \\ & Q^{-1} & \\ & & Q^{-1} \end{bmatrix} \begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} Q & & \\ & Q & \\ & & Q \end{bmatrix}$$

$$= \begin{bmatrix} A_2 & A_1 & A_0' \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $A_0' = Q^{-1} A_0 Q$.

□

Proof.

Similarly, we do the following computation to get B':

$$\begin{bmatrix} Q^{-1} & & \\ & Q^{-1} & \\ & & Q^{-1} \end{bmatrix} \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix} \begin{bmatrix} Q & & \\ & Q & \\ & & Q \end{bmatrix}$$

$$= \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix}.$$

Next we define $P = \begin{bmatrix} I_N & \mathbf{0} & \mathbf{0} \\ 0 & A_0' & A_1 \\ \mathbf{0} & \mathbf{0} & A_0' \end{bmatrix}$; Thus:

$$P(A' - \lambda B') = \begin{bmatrix} A_2 & A_1 & A_0' \\ A_1 & A_0' & \mathbf{0} \\ A_0' & \mathbf{0} & \mathbf{0} \end{bmatrix} - \lambda \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_1 & A_0' \\ \mathbf{0} & A_0' & \mathbf{0} \end{bmatrix}.$$

Proof.

Since A_3, A_2, and A_1 are symmetric, and A_0 is similar to a symmetric matrix with real entries, A' and B' are both similar to symmetric matrices with real entries, which implies that the BVP has all distinct real eigenvalues. □

An example to exhibit the validity of Theorem

Example

Given the the matrix of form of a BVP as:

$$\Gamma_\lambda \mathbf{y} = \begin{bmatrix} 3-\lambda & -1 & 0 & 0 \\ -1 & 3-\lambda & -1 & 0 \\ 0 & -1 & 3-\lambda & -1 \\ 0 & 0 & -5 & -1+2\lambda+5\lambda^2+\lambda^3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \mathbf{0}.$$

Reality of Eigenvalues

Example

Following the definitions for A_0, A_1, A_2, and A_3, we obtain

$$A = \left[\begin{array}{cccc|cccc|cccc} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 2 & 0 & 0 & -5 & -1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example

and

$$B = \left[\begin{array}{cccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]$$

Reality of Eigenvalues

Example

Therefore,

$$W = A^{-1}B = \begin{bmatrix} \mathbf{0} & I_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ -A_0^{-1}A_3 & -A_0^{-1}A_2 & -A_0^{-1}A_1 \end{bmatrix},$$

where

$$-A_0^{-1}A_3 = \frac{1}{61}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 21 \end{bmatrix}, \quad -A_0^{-1}A_2 = \frac{1}{61}\begin{bmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 40 \\ 0 & 0 & 0 & 105 \end{bmatrix},$$

and $\quad -A_0^{-1}A_1 = \frac{1}{61}\begin{bmatrix} 23 & 8 & 1 & 2 \\ 8 & 24 & 3 & 6 \\ 1 & 3 & 8 & 16 \\ -5 & -15 & -40 & 42 \end{bmatrix}.$

Example

The matrix W has 0 as an eigenvalue with multiplicity 6 and the following six nonzero real eigenvalues:

$$1.5251, \quad -0.9419, \quad -0.2238, \quad 0.6668, \quad 0.3371, \quad \text{and} \quad 0.2269.$$

The corresponding eigenvectors are

$$[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{v}_4 \;\; \mathbf{v}_5 \;\; \mathbf{v}_6] = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \vdots & \vdots & 0 & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Reality of Eigenvalues

Example

$$[\mathbf{v}_7 \quad \mathbf{v}_8 \quad \mathbf{v}_9 \quad \mathbf{v}_{10} \quad \mathbf{v}_{11} \quad \mathbf{v}_{12}] =$$

$$\begin{bmatrix} 0.0349 & -0.0100 & -0.0024 & -0.3505 & 0.6657 & -0.4903 \\ 0.0819 & -0.0407 & -0.0180 & -0.5258 & 0.0222 & 0.6902 \\ 0.1570 & -0.1553 & -0.1317 & -0.4384 & -0.6649 & -0.4813 \\ 0.2862 & -0.5900 & -0.9656 & -0.1319 & -0.0443 & -0.0126 \\ 0.0533 & 0.0094 & 0.0005 & -0.2337 & 0.2244 & -0.1112 \\ 0.1249 & 0.0383 & 0.0040 & -0.3506 & 0.0075 & 0.1566 \\ 0.2394 & 0.1463 & 0.0295 & -0.2923 & -0.2241 & -0.1092 \\ 0.4365 & 0.5557 & 0.2161 & -0.0880 & -0.0149 & -0.0029 \\ 0.0812 & -0.0089 & -0.0001 & -0.1558 & 0.0756 & -0.0252 \\ 0.1904 & -0.0361 & -0.0009 & -0.2338 & 0.0025 & 0.0355 \\ 0.3652 & -0.1378 & -0.0066 & -0.1949 & -0.0755 & -0.0248 \\ 0.6656 & -0.5234 & -0.0484 & -0.0586 & -0.0050 & -0.0006 \end{bmatrix}.$$

Certainly one can see that the eigenvalues and eigenvectors are all real.

Conclusion and Future Directions

- We describe the DSLBVP with a matrix equation ($\Gamma_\lambda \mathbf{y} = \mathbf{0}$) which involves the parameter λ in a cubic polynomial.

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- When A is nonsingular, we can formulate a process of reducing the DSLBVP into a regular eigenvalue problem so that many powerful existing tools of solving eigenvalue problems can be implemented.
- We give a condition on the boundary constraints under which all the eigenvalues are real.

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- Under what conditions will all the eigenvalues be distinct?
- Possibilities for further exploration also include investigating similar problems with higher degree boundary conditions.

Thank you for your Attention!