Using Matrix Pencils to Solve Discrete Sturm-Liouville Problems with Nonlinear Boundary Conditions

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Brief History

Sturm-Liouville Equation

- Named after Jacques Charles Franois Sturm (1803 – 1855) and Joseph Liouville (1809 – 1882).
- Second-order linear differential equation of the form

\[-\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = \lambda w(x)y.\]

- \(\lambda\) is a parameter.
- \(p(x), q(x)\) are positive real functions; \(w(x)\) is called a weight function (real).
- A solution pair \((\lambda, y)\) is called an eigen pair; \(\lambda\): eigenvalue; \(y\): eigenfunction corresponding to \(\lambda\).
Applications have appeared in science and engineering.

- Belinskiy and Dauer studied SLBVPs that arise in the study of waves of ice-covered oceans.
- Greenburg and Babuska investigated numerical solutions to SLBVPs that arise in acoustical problems.
- Belinskiy and Graef considered a nonlinear SLBVP that arises from the study of the torsion of a wing in a flow.
- Freiling and Yurko discussed inverse problems for a wave equation with a focused source of disturbance.
Continuous Sturm-Liouville Boundary Problem on \([0, b] \)

\[
\begin{align*}
Ly &= (1/r)(-py')' + qy = \lambda y \\
y(0) &= 0, \\
C(\lambda)y(b) &= D(\lambda)y'(b).
\end{align*}
\]
Converting the problem into Discrete Form

- Continuous Sturm-Liouville Boundary Problem on \([0, b]\)

\[
\begin{cases}
Ly = (1/r)(-py')' + qy = \lambda y \\
y(0) = 0, \\
C(\lambda)y(b) = D(\lambda)y'(b)).
\end{cases}
\]

- Let \(T = [0, t_1, \ldots, t_{N-1}, 1]\). Denote \(y_n\) for \(y(t_n)\) and \(h = t_{n+1} - t_n\).

Define

\[
\Delta y_n = \frac{y_{n+1} - y_n}{h} \quad \text{and} \quad \nabla y_n = \frac{y_n - y_{n-1}}{h}
\]
Converting the problem into Discrete Form

Continuous Sturm-Liouville Boundary Problem on \([0, b]\)

\[
\begin{align*}
L y &= (1/r)(-p y')' + q y = \lambda y \\
y(0) &= 0, \\
C(\lambda) y(b) &= D(\lambda) y'(b).
\end{align*}
\]

Let \(T = [0, t_1, \ldots, t_{N-1}, 1]\). Denote \(y_n\) for \(y(t_n)\) and \(h = t_{n+1} - t_n\). Define

\[
\Delta y_n = \frac{y_{n+1} - y_n}{h} \quad \text{and} \quad \nabla y_n = \frac{y_n - y_{n-1}}{h}
\]

The DSLBVP is of the following form, where \(L\) is the operator:

\[
\begin{align*}
L y_n &= \frac{1}{r_n} (\nabla(-p_n \Delta y_n) + q_n y_n) = \lambda y_n \\
y_0 &= 0 \\
C(\lambda) y_N &= D(\lambda)(-p_{N-1} \Delta y_{N-1}).
\end{align*}
\]
We simplify the operator as

$$Ly_n = -ay_{n+1} + \sigma y_n - ay_{n-1},$$

where $p_n = p, q_n = q, r_n = r$ are constants and $a = -\frac{p}{r^2}$, and $\sigma = 2\frac{p}{r^2} + q$.

When $C(\lambda)$ and $D(\lambda)$ are both linear in $\lambda$, we rewrite the boundary condition as $-(b_1 y_N + b_2 y_{N-1}) = \lambda (a_1 y_N + a_2 y_{N-1})$ and define

$$M = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}.$$

**Theorem (Linease case - Harmsen and Li, 2003)**

If $M$ is positive definite. Then:

1. All eigenvalues of the DSLBVP with linear BVC are real and simple.
2. The eigenvectors can be chosen to be real.
3. An Expansion Theory is given.
The boundary condition is

\[(c_0 + c_1\lambda + c_2\lambda^2) y_N = (d_0 + d_1\lambda + d_2\lambda^2) (-p_{N-1}\Delta y_{N-1}).\]

We then define

\[M = \begin{pmatrix} h & l \\ l & m \end{pmatrix},\]

where

\[-l = \begin{vmatrix} d_2 & d_0 \\ c_2 & c_0 \end{vmatrix}, \quad m = \begin{vmatrix} d_2 & d_1 \\ c_2 & c_1 \end{vmatrix}, \quad h = \begin{vmatrix} d_1 & d_0 \\ c_1 & c_0 \end{vmatrix}.\]

**Theorem (Quadratic case - Harmsen and Li, 2007)**

If \(M\) is positive definite. Then:

1. All eigenvalues of the DSLBVP with linear BVC are real and simple.
2. The eigenvectors can be chosen to be real.
3. An Expansion Theory is given.
We focus on DSLBVP with cubic nonlinearity in the BVC:

\[
\begin{align*}
L y_n &= -a y_{n+1} + \sigma y_n - a y_{n-1} = \lambda y \\
y(0) &= 0 \\
C(\lambda) y_N &= D(\lambda)(-p \Delta y_{N-1}).
\end{align*}
\]

We set:

\[C(\lambda) = c_0 + c_1 \lambda + c_2 \lambda^2 + c_3 \lambda^3\]
\[D(\lambda) = d_0 + d_1 \lambda + d_2 \lambda^2 + d_3 \lambda^3.\]

Define:

\[\alpha(\lambda) = \frac{p}{h} D(\lambda) = \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0\]
\[\beta(\lambda) = C(\lambda) + \frac{p}{h} D(\lambda) = \beta_3 \lambda^3 + \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0.\]

In particular, \[\alpha_0 = \frac{p}{h} d_0\]
and \[\beta_0 = c_0 + \alpha_0.\]
The matrix equation for the DSLBVP is:

$$\Gamma_\lambda \mathbf{y} = \mathbf{0}$$

where $$\mathbf{y}^T = (y_1, y_2, \ldots, y_N)$$.

$$\Gamma_\lambda = \begin{bmatrix}
\sigma - \lambda & -a & 0 & \cdots & \cdots & 0 \\
-a & \sigma - \lambda & -a & \cdots & \cdots & 0 \\
0 & -a & \sigma - \lambda & -a & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & -a & \sigma - \lambda & -a \\
0 & 0 & \cdots & 0 & \alpha(\lambda) & \beta(\lambda)
\end{bmatrix}$$

DSLBVP has nontrivial solutions if and only if

$$|\Gamma_\lambda| = 0.$$
Cubic Example

Example

For \( N = 4 \), \((p, q, r) = (1, 0, 16)\) and b.v.c.
\((-1 + 3\lambda + 2\lambda^3)y_4 = (-1 + 2\lambda^3)(-\Delta y_3).\) Thus
\(Ly_n = -y_{n+1} + 2y_n - y_{n-1} = \lambda y_n;\) we obtain:

\[
\begin{align*}
Ly_1 &= -y_2 + 2y_1 = \lambda y_1 \\
Ly_2 &= -y_3 + 2y_2 - y_1 = \lambda y_2 \\
Ly_3 &= -y_4 + 2y_3 - y_2 = \lambda y_3 \\
(-1 + 3\lambda + 2\lambda^3)y_4 &= (-1 + 2\lambda^3)(-2(y_4 - y_3)).
\end{align*}
\]
Example 2 Continued

From the equations, we obtain the matrix form:

\[
\Gamma_\lambda y = \begin{bmatrix}
2 - \lambda & -1 & 0 & 0 \\
-1 & 2 - \lambda & -1 & 0 \\
0 & -1 & 2 - \lambda & -1 \\
0 & 0 & 4 - 8\lambda^3 & -5 + 3\lambda + 10\lambda^3
\end{bmatrix} \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix} = 0.
\]

\[|\Gamma_\lambda| = 46\lambda - 56\lambda^2 + 39\lambda^3 - 71\lambda^4 + 52\lambda^5 - 10\lambda^6 - 8.\]
From the equations, we obtain the matrix form:

\[
\Gamma_\lambda y = \begin{bmatrix}
2 - \lambda & -1 & 0 & 0 \\
-1 & 2 - \lambda & -1 & 0 \\
0 & -1 & 2 - \lambda & -1 \\
0 & 0 & 4 - 8\lambda^3 & -5 + 3\lambda + 10\lambda^3
\end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = 0.
\]

\[
|\Gamma_\lambda| = 46\lambda - 56\lambda^2 + 39\lambda^3 - 71\lambda^4 + 52\lambda^5 - 10\lambda^6 - 8.
\]

Using \( \lambda = 3.2731 \), a root of the polynomial, we solve the matrix equation to obtain:

\[
\begin{align*}
y_1 &= 2.0712 \\
y_2 &= -2.6366 \\
y_3 &= 1.2855 \\
y_4 &= 1.
\end{align*}
\]
\[ \Gamma_\lambda = A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0, \quad \text{where:} \]

\[
A_3 = \begin{bmatrix}
0_{N-2} & 0 & 0 \\
0 & 0 & 0 \\
0 & \alpha_3 & \beta_3
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0_{N-2} & 0 & 0 \\
0 & 0 & 0 \\
0 & \alpha_2 & \beta_2
\end{bmatrix},
\]

\[
A_1 = \begin{bmatrix}
-I_{N-2} & 0 & 0 \\
0 & -1 & 0 \\
0 & \alpha_1 & \beta_1
\end{bmatrix}, \quad A_0 = \begin{bmatrix}
\sigma & a & 0 & \ldots & \ldots & 0 \\
a & \sigma & a & \ldots & \ldots & 0 \\
0 & a & \sigma & a & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & a & \sigma & a \\
0 & 0 & \ldots & 0 & \alpha_0 & \beta_0
\end{bmatrix}.
\]

\(\alpha_n\) and \(\beta_n\) are the coefficients of the \(n^{th}\) term of \(\alpha(\lambda)\) and \(\beta(\lambda)\) respectively.
The matrix equation:

\[ \Gamma_\lambda y = (A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0) y = 0 \]

has a nontrivial solution if and only if

\[ |A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0| = 0. \]

Moreover, \( \Gamma_\lambda y = 0 \) can be written as a matrix pencil equation:

\[
\begin{pmatrix}
\lambda^N y \\
\lambda^{N-1} y \\
\vdots \\
y
\end{pmatrix}
- \lambda B
\begin{pmatrix}
\lambda^N y \\
\lambda^{N-1} y \\
\vdots \\
y
\end{pmatrix} =
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix},
\]

which has non-trivial solutions if and only if

\[ |A - \lambda B| = 0. \]
**Matrix Pencil**

**Definition**

The expression $A - \lambda B$, where $A$ and $B$ are $m \times n$ matrices, is called a matrix pencil, or just pencil. Here $\lambda$ is an indeterminate, not a particular numerical value.

Thus eigenvalues of the DSLBVP are the eigenvalues of the pencil $A - \lambda B$ where

$$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ 0 & I_N & 0 \\ I_N & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -A_3 & 0 & 0 \\ 0 & 0 & I_N \\ 0 & I_N & 0 \end{bmatrix}.$$

Thus we solve

$$\begin{bmatrix} A_2 & A_1 & A_0 \\ 0 & I_N & 0 \\ I_N & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} -A_3 & 0 & 0 \\ 0 & 0 & I_N \\ 0 & I_N & 0 \end{bmatrix} \begin{bmatrix} \lambda^2 y \\ \lambda y \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
The construction of $A$ and $B$ comes from the following lemma:

**Lemma**

The $N^{th}$ degree matrix polynomial equation:

$$(A_N \lambda^N + A_{N-1} \lambda^{N-1} + \cdots + A_0)y = 0,$$

is equivalent to

$$(A - \lambda B) \begin{bmatrix} \lambda^N y \\ \lambda^{N-1} y \\ \vdots \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

where

$$A = \begin{bmatrix} A_{N-1} & \cdots & A_1 & A_0 \\ \vdots & \ddots & \vdots & \vdots \\ I_N & \cdots & I_N & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -A_N & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_N & 0 \end{bmatrix}. $$
Assume $A$ is invertible and $\lambda$ is not equal to zero, from the matrix pencil we have:

$$|A - \lambda B| = 0 \iff \left| \frac{1}{\lambda} A^{-1} (A - \lambda B) \right| = 0 \iff \left| \frac{1}{\lambda} I - A^{-1} B \right| = 0 \iff \left| A^{-1} B - \mu I \right| = 0,$$

where $\mu = \frac{1}{\lambda}$, which is a regular eigenvalue problem (REP).

Obviously, we need $A$ to be invertible in order to be able to solve the DSLBVP as a REP.
Conditions for $A_0$ to be Nonsingular

Recall that,

$$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ 0 & I_N & 0 \\ I_N & 0 & 0 \end{bmatrix},$$

and $A_0$ lies on the minor diagonal of the block matrix $A$ and all other blocks on the minor diagonal of $A$ are $I_N$. Thus $|A| = \pm |A_0| \Rightarrow A$ is invertible if and only if $A_0$ is invertible.

Recall that $A_0$ is a tridiagonal matrix of the form:

$$A_0 = \begin{bmatrix} \sigma & a & 0 & \cdots & 0 \\ a & \sigma & a & \cdots & 0 \\ 0 & a & \sigma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma \end{bmatrix},$$

where $\alpha_0$ and $\beta_0$ are the constant terms of $\alpha(\lambda)$ and $\beta(\lambda)$ respectively.
Recall that,

\[
A = \begin{bmatrix}
A_2 & A_1 & A_0 \\
0 & I_N & 0 \\
I_N & 0 & 0
\end{bmatrix},
\]

- $A_0$ lies on the minor diagonal of the block matrix $A$ and all other blocks on the minor diagonal of $A$ are $I_N$. 

Recall that $A_0$ is a tridiagonal matrix of the form:

\[
A_0 = \begin{bmatrix}
\sigma & a_0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & a_{N-1} & \sigma & a_N \\
\alpha_0 & \beta_0 & \cdots & \alpha_N
\end{bmatrix},
\]

where $\alpha_0$ and $\beta_0$ are the constant terms of $\alpha(\lambda)$ and $\beta(\lambda)$ respectively.
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- Thus $|A| = \pm |A_0| \implies A$ is invertible if and only if $A_0$ is invertible.
Conditions for $A_0$ to be Nonsingular

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\[ A = \begin{bmatrix} A_2 & A_1 & A_0 \\ 0 & I_N & 0 \\ I_N & 0 & 0 \end{bmatrix}, \]

- $A_0$ lies on the minor diagonal of the block matrix $A$ and all other blocks on the minor diagonal of $A$ are $I_N$.
- Thus $|A| = \pm |A_0| \implies A$ is invertible if and only if $A_0$ is invertible.
- Recall that $A_0$ is a tridiagonal matrix of the form:

\[ A_0 = \begin{bmatrix} \sigma & a & 0 & \cdots & \cdots & 0 \\ a & \sigma & a & \cdots & \cdots & 0 \\ 0 & a & \sigma & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & \sigma & a \\ 0 & 0 & \cdots & 0 & \alpha_0 & \beta_0 \end{bmatrix}, \]

where $\alpha_0$ and $\beta_0$ are the constant terms of $\alpha(\lambda)$ and $\beta(\lambda)$ respectively.
Conditions for $A_0$ to be Nonsingular

**Lemma (Mikkawy and Karawia)**

Given the general tridiagonal matrix:

$$T_n = \begin{bmatrix}
\sigma_1 & a_1 & 0 & \cdots & \cdots & 0 \\
b_2 & \sigma_2 & a_2 & \cdots & \cdots & 0 \\
0 & b_3 & \sigma_3 & a_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & b_{n-1} & \sigma_{n-1} & a_{n-1} \\
0 & 0 & \cdots & 0 & b_n & \sigma_n
\end{bmatrix},$$

where $a_1a_2\cdots a_{n-1} \neq 0$ and $b_2b_3\cdots b_{n-1} \neq 0$.

$$|T_i| = \begin{cases} 
1 & \text{if } i = 0 \\
\sigma_1 & \text{if } i = 1 \\
\sigma_i T_{i-1} - b_i T_{i-1} T_{i-2} & \text{if } i = 2, 3, \cdots, n.
\end{cases}$$
**Lemma (1)**

Given the \((N \times N)\) tridiagonal matrix \(A_0\) as above. Let \(U_i\) be the determinant of the \(i \times i\) main diagonal of \(A_0\), that is \(U_i\) is of the form

\[
U_i = \begin{vmatrix}
\sigma & a & 0 & \cdots & \cdots & 0 \\
a & \sigma & a & \cdots & \cdots & 0 \\
0 & a & \sigma & a & \cdots & 0 \\
0 & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & a & \sigma & a \\
0 & 0 & \cdots & 0 & a & \sigma
\end{vmatrix}
\]

\[0 < i \leq N - 1.\]

Let \(|A_0| = U_N\), then \(U_N\) is given iteratively as

\[
\begin{cases}
U_0 = 1 \\
U_1 = \sigma \\
U_i = \sigma U_{i-1} - a^2 U_{i-2} \quad \text{for } 1 < i < N \\
U_N = \beta_0 U_{N-1} - a\alpha_0 U_{N-2}.
\end{cases}
\]
Finding Conditions where $A_0$ is Nonsingular

From Lemma 1

$U_i = \sigma U_{i-1} - a^2 U_{i-2}$

We solve

$x^2 - \sigma x + a^2 = 0$ for the roots $s_1$ and $s_2$. 

$s_1 \neq s_2$:

$U_i = \frac{(s_2 - s_1) s_i}{s_1} + \frac{(s_2 s_1 - s_1) s_i}{s_2}$, $0 \leq i < N$

$s_1 = s_2$:

$U_i = s_i, 0 \leq i < N$.
Finding Conditions where $A_0$ is Nonsingular

From Lemma 1

$$U_i = \sigma U_{i-1} - a^2 U_{i-2}$$

We solve

$$x^2 - \sigma x + a^2 = 0$$

for the roots $s_1$ and $s_2$.

- $s_1 \neq s_2$: $U_i = \left( \frac{-s_1}{s_2 - s_1} \right) s_1^i + \left( \frac{s_2}{s_2 - s_1} \right) s_2^i, \quad 0 \leq i < N$
Finding Conditions where $A_0$ is Nonsingular

From Lemma 1

$$U_i = \sigma U_{i-1} - a^2 U_{i-2}$$

We solve

$$x^2 - \sigma x + a^2 = 0$$

for the roots $s_1$ and $s_2$.

- $s_1 \neq s_2$: $U_i = \left(\frac{-s_1}{s_2-s_1}\right)s_1^i + \left(\frac{s_2}{s_2-s_1}\right)s_2^i, \quad 0 \leq i < N$

- $s_1 = s_2$: $U_i = s_1^i + is_2^i, \quad 0 \leq i < N$
**Theorem**

Let $U_N = |A_0|$ and $s_1, s_2$ be the solutions to the equation $x^2 - \sigma x + a^2 = 0$.

1. In the case $\sigma^2 > 4a^2$, we have $s_1 > s_2 > 0$,

$$U_N = \frac{1}{s_1 - s_2} \left[ \left( \beta_0 - \frac{\alpha_0}{a} s_2 \right) s_1^N - \left( \beta_0 - \frac{\alpha_0}{a} s_1 \right) s_2^N \right]. \quad (1)$$

2. In the case $\sigma^2 = 4a^2$, we have $s_1 = s_2$,

$$U_N = \beta_0 N \left( \frac{\sigma}{2} \right)^{N-1} - \alpha_0 a (N - 1) \left( \frac{\sigma}{2} \right)^{N-2}. \quad (2)$$
**Remark**

Assuming $\alpha_0 = 0$ and $\beta_0 \neq 0$, we note that the tridiagonal matrix $A_0$ is nonsingular for both cases where $s_1 = s_2$ and $s_1 \neq s_2$. The reason is that when $s_1 \neq s_2$, $U_N = \frac{\beta_0(s_1^N - s_2^N)}{s_1 - s_2}$ is not equal to zero since $\beta_0 \neq 0$. And in the case where $s_1 = s_2$, $U_N = \beta_0 N \left(\frac{\sigma}{2}\right)^{N-1}$ is not equal to zero since $\beta_0 \neq 0$.

**Theorem**

Consider the $(N \times N)$ tridiagonal matrix $A_0$ as above with $\alpha_0$ not equal to zero. Let $U_N = |A_0|$ and $(\alpha_0, \beta_0) \neq (0, 0)$. Then $U_N$ is not equal to zero in any of the following cases:

1. $s_1 = s_2$ or
2. $s_1 \neq s_2$ but $\beta_0 - \frac{\alpha_0}{a} s_1 = 0$ or
3. $s_1 \neq s_2$ but $\beta_0 - \frac{\alpha_0}{a} s_1 \neq 0$ and $\frac{\beta_0}{\alpha_0} < \frac{s_2}{a}$. 

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$s_1 = s_2$.

Case 1: $\beta_0 = 0$. Obvious.

Case 2: $\beta_0 \neq 0$. $U_N = 0$ implies

$$N = \frac{\alpha_0}{\beta_0 + \alpha_0},$$

A contradiction since $N$ is a positive integer. Therefore, $U_N \neq 0$.  

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Using Matrix Pencils to Solve Discrete Sturm-Liouville Problems
**Proof.**

(Sketch)

We have two cases to consider when \( s_1 \neq s_2 \).

**Case 1:** \( \beta_0 - \frac{\alpha_0}{a} s_1 = 0 \iff \beta_0 - \frac{\alpha_0}{a} s_2 \neq 0 \). Obvious.

**Case 2:** \( \beta_0 - \frac{\alpha_0}{a} s_1 \neq 0 \) and \( \frac{\beta_0}{\alpha_0} < \frac{s_2}{a} \)

\[
U_N = 0 \iff \left( \frac{s_1}{s_2} \right)^N = \frac{\beta_0 - \frac{\alpha_0}{a} s_1}{\beta_0 - \frac{\alpha_0}{a} s_2} = \frac{a\beta_0 - \alpha_0 s_1}{a\beta_0 - \alpha_0 s_2}.
\]

Further investigation shows that the given condition implies \( s_1 < s_2 \). This contradicts the fact that

\[
s_1 = \frac{\sigma + \sqrt{\sigma^2 - 4a^2}}{2} > s_2 = \frac{\sigma - \sqrt{\sigma^2 - 4a^2}}{2} > 0.
\]
The Form of DSLBVP as REP

With the results on the non-singularity of $A_0$ established, we can proceed to discuss $|A^{-1}B - \mu I|$. We have

$$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ 0 & I_N & 0 \\ I_N & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -A_3 & 0 & 0 \\ 0 & 0 & I_N \\ 0 & I_N & 0 \end{bmatrix}.$$ 

thus

$$A^{-1} = \begin{bmatrix} 0 & 0 & I_N \\ 0 & I_N & 0 \\ A_0^{-1} & -A_0^{-1}A_1 & -A_0^{-1}A_2 \end{bmatrix}.$$ 

$$W = A^{-1}B = \begin{bmatrix} 0 & I_N & 0 \\ 0 & 0 & I_N \\ -A_0^{-1}A_3 & -A_0^{-1}A_2 & -A_0^{-1}A_1 \end{bmatrix}.$$ 

We solve $|W - \mu I| = 0$ to determine the solution of the DSLBVP.
We first show that $A_0$ is similar to a symmetric matrix.

**Lemma (4)**

If $a\alpha_0 > 0$, then $A_0$ is similar to a symmetric matrix.

**Proof.**

Put $A_0$ in the form

$$A_0 = \begin{bmatrix} E_1 & E_2 \\ E_3 & \beta_0 \end{bmatrix},$$

where

$E_1(N-1)\times(N-1)$ — major diagonal sub-matrix of $A_0$,

$E_2 = [0 \; \cdots \; 0 \; a]^T$, and $E_3 = [0 \; \cdots \; 0 \; \alpha_0]$. 
Reality of Eigenvalues

**Proof.**

Define

\[ Q = \begin{bmatrix} I_{N-1} & 0 \\ 0 & \sqrt{\frac{\alpha_0}{a}} \end{bmatrix}, \]

then

\[ Q^{-1} A_0 Q = \begin{bmatrix} E_1 & \sqrt{\frac{\alpha_0}{a}} E_2 \\ \sqrt{\frac{a}{\alpha_0}} E_3 & \beta_0 \end{bmatrix} = A'_0. \]

Obviously \( E_1 \) is symmetric, and \( \left( \sqrt{\frac{\alpha_0}{a}} E_2 \right)^T = \begin{bmatrix} 0 & \cdots & 0 & \sqrt{\alpha_0 a} \end{bmatrix} = \sqrt{\frac{a}{\alpha_0}} E_3. \)

Therefore \( A_0 \) is similar to the symmetric matrix to which we denote \( A'_0. \)
We show an example here to demonstrate Lemma 4.

**Example**

Take

\[
G_2 = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \text{ which implies } E_1 = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix},
\]

\[
E_2 = [0 \ 0 \ -4]^T, \text{ and } E_3 = [0 \ 0 \ -2]. \text{ Therefore }
\]

\[
Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ and } Q^{-1}G_2Q = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -2\sqrt{2} \\ 0 & 0 & -2\sqrt{2} & 3 \end{bmatrix}
\]

which is obviously symmetric.
Reality of Eigenvalues

**Theorem**

If \( D(\lambda) \equiv d_0 < 0 \), then all the eigenvalues of the DSLBVP are real.

**Proof.**

The eigenvalues of the DSLBVP are the eigenvalues of the matrix pencil \( A - \lambda B \). We know from Linear Algebra that if \( A \) and \( B \) are symmetric matrices with real entries, then the eigenvalues of the pencil are all real. Given

\[
D(\lambda) \equiv d_0 < 0, \quad \text{then} \quad \alpha(\lambda) \equiv \frac{pN-1}{h}d_0 = \alpha_0 < 0.
\]

- \( h \) and \( p \) are both positive
- \( a\alpha_0 > 0 \), we can apply Lemma 4 to make \( A_0 \) similar to a symmetric matrix
- \( D(\lambda) = d_0 \) implies that \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \).
- Thus \( A_1, A_2, A_3 \) are all diagonal and symmetric matrices.
**Proof.**

Take our

\[ Q = \begin{bmatrix}
    I_{N-1} & 0 \\
    0 & \sqrt{\frac{\alpha_0}{a}}
\end{bmatrix}, \]

and make \( A \) similar to a symmetric matrix denoted \( A' \) by computing:

\[
\begin{bmatrix}
    Q^{-1} & 0 \\
    0 & Q^{-1}
\end{bmatrix}
\begin{bmatrix}
    A_2 & A_1 & A_0 \\
    0 & I_N & 0 \\
    I_N & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    Q^{-1} & 0 \\
    0 & Q^{-1}
\end{bmatrix}
= \begin{bmatrix}
    A_2 & A_1 & A'_0 \\
    0 & I_N & 0 \\
    I_N & 0 & 0
\end{bmatrix},
\]

where \( A'_0 = Q^{-1}A_0Q \).
Proof.

Similarly, we do the following computation to get $B'$:

\[
\begin{bmatrix}
Q^{-1} & Q^{-1} & Q^{-1} & Q^{-1}
\end{bmatrix}
\begin{bmatrix}
-A_3 & 0 & 0 & 0
0 & 0 & I_N & 0
0 & I_N & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Q & Q
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-A_3 & 0 & 0 & 0
0 & 0 & I_N & 0
0 & I_N & 0 & 0
\end{bmatrix}.
\]

Next we define $P = \begin{bmatrix}
l_N & 0 & 0 & 0
0 & A'_0 & A_1 & 0
0 & 0 & A'_0 & 0
\end{bmatrix}$; Thus:

\[
P(A' - \lambda B') = \begin{bmatrix}
A_2 & A_1 & A'_0 & 0
A_1 & A'_0 & 0 & 0
A'_0 & 0 & 0 & 0
\end{bmatrix}
- \lambda \begin{bmatrix}
-A_3 & 0 & 0 & 0
0 & A_1 & A'_0 & 0
0 & A'_0 & 0 & 0
\end{bmatrix}.
\]
REALITY OF EIGENVALUES

**Proof.**

Since $A_3$, $A_2$, and $A_1$ are symmetric, and $A_0$ is similar to a symmetric matrix with real entries, $A'$ and $B'$ are both similar to symmetric matrices with real entries, which implies that the BVP has all distinct real eigenvalues.

An example to exhibit the validity of Theorem

**Example**

Given the matrix of form of a BVP as:

$$
\begin{bmatrix}
3 - \lambda & -1 & 0 & 0 \\
-1 & 3 - \lambda & -1 & 0 \\
0 & -1 & 3 - \lambda & -1 \\
0 & 0 & -5 & -1 + 2\lambda + 5\lambda^2 + \lambda^3 
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 
\end{bmatrix} = 0.
$$

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Using Matrix Pencils to Solve Discrete Sturm-Liouville Problems
Following the definitions for $A_0$, $A_1$, $A_2$, and $A_3$, we obtain

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\
0 & 0 & 0 & 5 & 0 & 0 & 0 & 2 & 0 & 0 & -5 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
### Example

and

$$B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$
\[ W = A^{-1}B = \begin{bmatrix}
0 & I_N & 0 \\
0 & 0 & I_N \\
-A_0^{-1}A_3 & -A_0^{-1}A_2 & -A_0^{-1}A_1
\end{bmatrix}, \]

where

\[ -A_0^{-1}A_3 = \frac{1}{61} \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 8 \\
0 & 0 & 0 & 21
\end{bmatrix}, \quad -A_0^{-1}A_2 = \frac{1}{61} \begin{bmatrix}
0 & 0 & 0 & 5 \\
0 & 0 & 0 & 15 \\
0 & 0 & 0 & 40 \\
0 & 0 & 0 & 105
\end{bmatrix}, \]

and \[ -A_0^{-1}A_1 = \frac{1}{61} \begin{bmatrix}
23 & 8 & 1 & 2 \\
8 & 24 & 3 & 6 \\
1 & 3 & 8 & 16 \\
-5 & -15 & -40 & 42
\end{bmatrix}. \]
**Example**

The matrix $W$ has 0 as an eigenvalue with multiplicity 6 and the following six nonzero real eigenvalues:

$$1.5251, \ -0.9419, \ -0.2238, \ 0.6668, \ 0.3371, \ \text{and} \ 0.2269.$$  

The corresponding eigenvectors are

$$[v_1 \ v_2 \ v_3 \ v_4 \ v_5 \ v_6] = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
Certainly one can see that the eigenvalues and eigenvectors are all real.
Conclusion and Future Directions

- We describe the DSLBVP with a matrix equation $(\Gamma_\lambda y = 0)$ which involves the parameter $\lambda$ in a cubic polynomial.
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We describe the DSLBVP with a matrix equation \( (\Gamma_{\lambda}y = 0) \) which involves the parameter \( \lambda \) in a cubic polynomial.

We then construct a new matrix equation \( (A - B\lambda)y = 0 \) which has the same solution space.

When \( A \) is nonsingular, we can formulate a process of reducing the DSLBVP into a regular eigenvalue problem so that many powerful existing tools of solving eigenvalue problems can be implemented.

We give a condition on the boundary constraints under which all the eigenvalues are real.
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- Using Matrix Pencils to Solve Discrete Sturm-Liouville Problems

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What can we say when the matrix $A$ is singular?

Can we give less restricted conditions to guarantee the reality of the eigenvalues of the problem?

Under what conditions will all the eigenvalues be distinct?

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Thank you for your Attention!