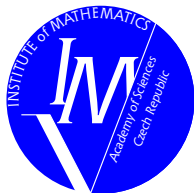


# On boundary discrete maximum principle in the finite element method

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- ▶ Monotone matrix

$$Az = b$$

$$(b \geq 0 \Rightarrow z \geq 0) \Leftrightarrow A^{-1} \geq 0$$

- ▶ Discrete conservation of nonnegativity (DCN)

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

$$(f(x) \geq 0 \Rightarrow u_h(x) \geq 0) \Leftrightarrow G_h^0(x, y) \geq 0$$



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## Maximum principle – strong sense

- ▶ Classical formulation:

$$\begin{aligned} -\operatorname{div}(\mathcal{A}\nabla u) + \mathbf{b} \cdot \nabla u + cu &= f && \text{in } \Omega \\ u &= g_D && \text{on } \Gamma_D \\ \alpha u + (\mathcal{A}\nabla u) \cdot \mathbf{n} &= g_N && \text{on } \Gamma_N \end{aligned}$$

- ▶  $\Omega \subset \mathbb{R}^d$ ,  $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$
- ▶  $\boldsymbol{\xi}^T \mathcal{A}(\mathbf{x}) \boldsymbol{\xi} \geq \lambda_{\min} |\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d, \forall \mathbf{x} \in \Omega$
- ▶  $c - \frac{1}{2} \operatorname{div} \mathbf{b} \geq 0 \quad \text{in } \Omega \quad \text{and} \quad \alpha + \frac{1}{2} \mathbf{b} \cdot \mathbf{n} \geq 0 \quad \text{on } \Gamma_N$



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- ▶ Maximum principle (MP):

$$f \leq 0 \text{ and } g_N \leq 0 \quad \Rightarrow \quad \max_{\Omega} u \leq \max_{\Gamma_D} \max\{0, u\}$$

- ▶ Conservation of nonnegativity (CN):

$$f \geq 0, \quad g_D \geq 0, \quad \text{and } g_N \geq 0 \quad \Rightarrow \quad u \geq 0$$

- ▶ **Theorem:** If  $c \geq 0$  and  $\alpha \geq 0$  then  $\text{MP} \Leftrightarrow \text{CN}$

## Maximum principle – weak sense

- ▶ Weak formulation:  $u = u^0 + u^\partial$

$$u^0 \in V^0 : \quad a(u^0, v) = F(v) - a(u^\partial, v) \quad \forall v \in V^0$$

- ▶  $u^\partial \in H^1(\Omega)$ ,  $u^\partial = g_D$  on  $\Gamma_D$

- ▶  $V^0 = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$

- ▶  $a(u, v) = \int_{\Omega} (\mathcal{A} \nabla u) \cdot \nabla v + (\mathbf{b} \cdot \nabla u)v + cuv \, dx + \int_{\Gamma_N} \alpha uv \, ds$

- ▶  $F(v) = \int_{\Omega} fv \, dx + \int_{\Gamma_N} g_N v \, ds$



## Maximum principle – weak sense

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- ▶ Maximum principle (MP):

$$\begin{aligned} f \leq 0 \text{ a.e. in } \Omega \text{ and } g_N \leq 0 \text{ a.e. on } \Gamma_N \\ \Rightarrow \quad \operatorname{ess\,sup}_{\bar{\Omega}} u \leq \operatorname{ess\,sup}_{\Gamma_D} \max\{0, u\} \end{aligned}$$

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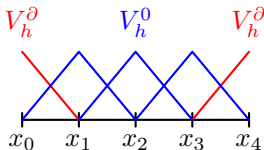
- ▶ **Theorem:** If  $c \geq 0$  and  $\alpha \geq 0$  then  $\text{MP} \Leftrightarrow \text{CN}$

# Finite element method (FEM)

- ▶ *hp*-FEM:  $u_h = u_h^0 + u_h^\partial$

$$u_h^0 \in V_h^0 : a(u_h^0, v_h^0) = F(v_h^0) - a(u_h^\partial, v_h^0) \quad \forall v_h^0 \in V_h^0$$

- ▶  $\mathcal{T}_h$  triangulation of  $\Omega$
- ▶  $p_K$  polynomial degree on  $K \in \mathcal{T}_h$
- ▶  $X_h = \{v_h \in H^1(\Omega) : v_h|_K \in P^{p_K}(K), K \in \mathcal{T}_h\}$
- ▶  $V_h^0 = X_h \cap V^0$
- ▶  $X_h = V_h^0 \oplus V_h^\partial$
- ▶  $u_h^\partial \in V_h^\partial, \quad u_h^\partial \approx g_D$  on  $\Gamma_D$





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▶  $G_{h,y}^0 \in V_h^0 : a(v_h^0, G_{h,y}^0) = v_h^0(y) \quad \forall v_h^0 \in V_h^0, y \in \Omega$

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▶  $G_{h,y}^\partial \in V_h^\partial : \int_{\Gamma_D} v_h^\partial(s) G_{h,y}^\partial(s) ds = v_h^\partial(y) - a(v_h^\partial, G_{h,y}^0)$   
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 $= F(G_{h,y}^0) + \int_{\Gamma_D} g_D(s) G_{h,y}^\partial(s) ds - u_h^\partial(y)$

- ▶  $u_h(y) = \int_{\Omega} f G_{h,y}^0 dx + \int_{\Gamma_N} g_N G_{h,y}^0 ds + \int_{\Gamma_D} g_D G_{h,y}^\partial ds$



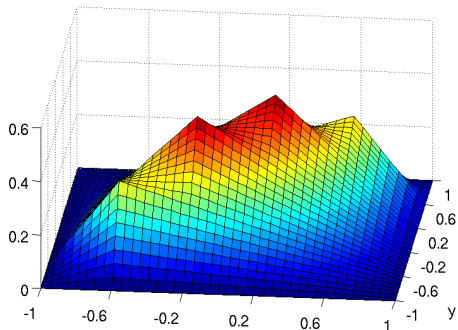


# Examples of $G_{h,y}^0$ and $G_{h,y}^\partial$

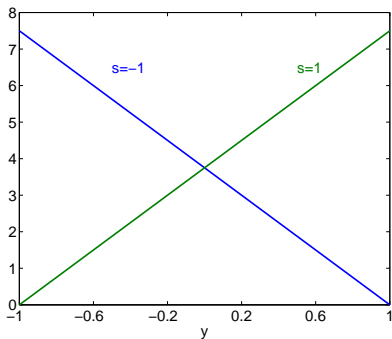
$$-u'' = f \quad \text{in } (-1, 1),$$

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$$G_{h,y}^0(x)$$



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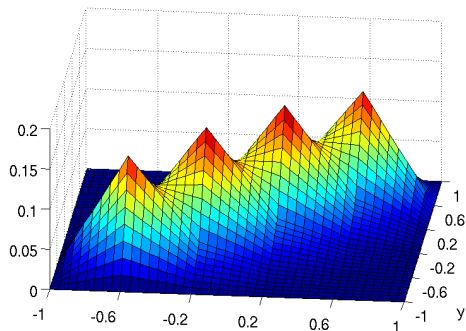


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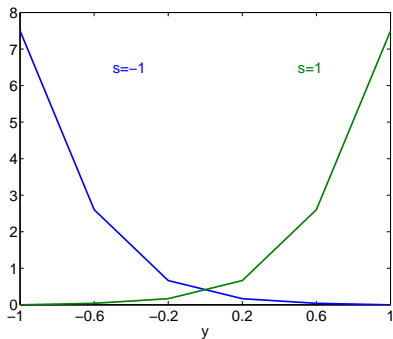
$$-u'' + 10u = f \quad \text{in } (-1, 1),$$

$$u(-1) = u(1) = g_D$$

$$G_{h,y}^0(x)$$



$$G_{h,y}^\partial(s)$$





# Maximum principle

- ▶ Discrete maximum principle (DMP):

$f \leq 0$  a.e. in  $\Omega$  and  $g_N \leq 0$  a.e. on  $\Gamma_N$

$$\Rightarrow \max_{\bar{\Omega}} u_h \leq \max_{\Gamma_D} \max\{0, u_h\}$$

- ▶ Discrete conservatin of nonnegativity (DCN):

$f \geq 0$  a.e. in  $\Omega$ ,  $g_D \geq 0$  a.e. on  $\Gamma_D$ , and  $g_N \geq 0$  a.e. on  $\Gamma_N$

$$\Rightarrow u_h \geq 0 \text{ in } \Omega$$

## Theorem (main)

$DCN \Leftrightarrow (a) G_{h,y}^0(x) \geq 0 \quad \forall (x,y) \in \Omega^2$

$(b) G_{h,y}^\partial(s) \geq 0 \quad \forall (s,y) \in \Gamma_D \times \Omega$

Proof:

$$u_h(y) = \int_{\Omega} f G_{h,y}^0 dx + \int_{\Gamma_N} g_N G_{h,y}^0 ds + \int_{\Gamma_D} g_D G_{h,y}^\partial ds$$



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But:

(i) (DCN)  $\not\Rightarrow$  (DMP)

(ii)  $G_{h,y}^\partial \not\geq 0$  up to exceptional cases

# Expressions for the DGF



- ▶  $\varphi_1^0, \varphi_2^0, \dots, \varphi_{N^0}^0$  basis in  $V_h^0$
- ▶  $\varphi_1^\partial, \varphi_2^\partial, \dots, \varphi_{N^\partial}^\partial$  basis in  $V_h^\partial$
- ▶  $A \in \mathbb{R}^{N^0 \times N^0}$ ,  $A_{ij} = a(\varphi_j^0, \varphi_i^0) \quad i, j = 1, 2, \dots, N^0$
- ▶  $A^\partial \in \mathbb{R}^{N^0 \times N^\partial}$ ,  $A_{ik}^\partial = a(\varphi_k^\partial, \varphi_i^0) \quad i = 1, \dots, N^0, k = 1, \dots, N^\partial$
- ▶  $M^\partial \in \mathbb{R}^{N^\partial \times N^\partial}$ ,  $M_{kl}^\partial = \int_{\Gamma_D} \varphi_\ell^\partial \varphi_k^\partial ds \quad k, \ell = 1, 2, \dots, N^\partial$
- ▶  $G_{h,y}^0(x) = \sum_{i=1}^{N^0} \sum_{j=1}^{N^0} \varphi_i^0(y) (A^{-1})_{ij} \varphi_j^0(x)$
- ▶  $G_{h,y}^\partial(s) = \sum_{k=1}^{N^\partial} \sum_{\ell=1}^{N^\partial} \varphi_k^\partial(s) (M^\partial)^{-1}_{kl} \left[ \varphi_\ell^\partial(y) - \sum_{i=1}^{N^0} \sum_{j=1}^{N^0} \varphi_i^0(y) (A^{-1})_{ij} A_{j\ell}^\partial \right]$



## Maximum principle – remedy

- ▶ Discrete maximum principle (DMP):

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$$\Rightarrow u_h \geq 0 \text{ in } \Omega$$

### Theorem (main)

$DCN \Leftrightarrow (a) G_{h,y}^0(x) \geq 0 \quad \forall (x,y) \in \Omega^2$

$(b) (I - \Pi_h^0)u_h^\partial \geq 0 \quad \forall u_h^\partial \in V_h^\partial, u_h^\partial \geq 0 \text{ in } \Omega$

Elliptic projection:  $\Pi_h^0 : X_h \mapsto V_h^0$

$$\Pi_h^0 w_h \in V_h^0 : a(w_h - \Pi_h^0 w_h, v_h^0) = 0 \quad \forall v_h^0 \in V_h^0$$



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Proof:

$$u_h(y) = \int_{\Omega} f(x) G_{h,y}^0(x) dx + \int_{\Gamma_N} g_N(s) G_{h,y}^0(s) ds + (I - \Pi_h^0)u_h^\partial(y)$$



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$(b) (I - \Pi_h^0)u_h^\partial \geq 0 \quad \forall u_h^\partial \in V_h^\partial, u_h^\partial \geq 0 \text{ in } \Omega$

Now:

(i)  $DCN \Leftrightarrow DMP$

(ii) Condition (b) can be satisfied in nontrivial cases.



## Theorem

Let the basis functions be such that

$$\sum_{i=1}^{N^0} c_i^0 \varphi_i^0(x) \geq 0 \quad \forall x \in \Omega \quad \Leftrightarrow \quad c_i^0 \geq 0 \quad \forall i = 1, 2, \dots, N^0,$$

$$\sum_{\ell=1}^{N^\partial} c_\ell^\partial \varphi_\ell^\partial(x) \geq 0 \quad \forall x \in \Omega \quad \Leftrightarrow \quad c_\ell^\partial \geq 0 \quad \forall \ell = 1, 2, \dots, N^\partial.$$

Then DCN holds true if and only if

$$(a) \quad A^{-1} \geq 0 \quad \text{and} \quad (b) \quad -A^{-1}A^\partial \geq 0.$$

Proof:

$$(a) \quad G_{h,y}^0(x) = \sum_{i=1}^{N^0} \sum_{j=1}^{N^0} \varphi_i^0(y) (A^{-1})_{ij} \varphi_j^0(x) \geq 0 \quad \Leftrightarrow \quad (A^{-1})_{ij} \geq 0$$

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$$\sum_{\ell=1}^{N^\partial} c_\ell^\partial \varphi_\ell^\partial(x) \geq 0 \quad \forall x \in \Omega \quad \Leftrightarrow \quad c_\ell^\partial \geq 0 \quad \forall \ell = 1, 2, \dots, N^\partial.$$

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$$(a) \quad A^{-1} \geq 0 \quad \text{and} \quad (b) \quad -A^{-1}A^\partial \geq 0.$$

Proof:

$$(b) \quad u_h^\partial(y) = \sum_{k=1}^{N^\partial} c_k^\partial \varphi_k^\partial(y)$$

$$(I - \Pi_h^0)u_h^\partial(y) = \sum_{\ell=1}^{N^\partial} c_\ell^\partial \left[ \varphi_\ell^\partial(\mathbf{y}) - \sum_{i=1}^{N^0} \sum_{j=1}^{N^0} \varphi_i^0(\mathbf{y}) (A^{-1})_{ij} A_{j\ell}^\partial \right]$$

## Theorem

Let the basis functions be such that

$$\sum_{i=1}^{N^0} c_i^0 \varphi_i^0(x) \geq 0 \quad \forall x \in \Omega \quad \Leftrightarrow \quad c_i^0 \geq 0 \quad \forall i = 1, 2, \dots, N^0,$$

$$\sum_{\ell=1}^{N^\partial} c_\ell^\partial \varphi_\ell^\partial(x) \geq 0 \quad \forall x \in \Omega \quad \Leftrightarrow \quad c_\ell^\partial \geq 0 \quad \forall \ell = 1, 2, \dots, N^\partial.$$

Then DCN holds true if and only if

$$(a) \quad A^{-1} \geq 0 \quad \text{and} \quad (b) \quad -A^{-1}A^\partial \geq 0.$$

## Remarks:

- ▶ Conditions (a) and (b) for finite differences  
are due to P.G. Ciarlet, 1970.
- ▶ If  $a(\varphi_i, \varphi_j) \leq 0$  for  $i \neq j$  then  $A^{-1} \geq 0$  and  $A^\partial \leq 0$   
( $A$  is M-matrix).



- ▶ Further modification of the definition of the DCN  $f \mapsto f_h$   
 $g^N \mapsto g_{Nh}$

- ▶  $\mathcal{T}_h$  fixed ( $V_h^0$  fixed) and want  $f \geq 0, \dots \Rightarrow u_h \geq 0$

*In the language of matrices:*

Want  $x = A^{-1}b \geq 0$  for all  $b \geq 0$  (i.e. want  $A^{-1} \geq 0$ )

- ▶  $f \geq 0$  fixed and want  $\mathcal{T}_h$  such that  $u_h \geq 0$

*In the language of matrices:*

Given  $b \geq 0$  construct  $A$  such that  $x = A^{-1}b \geq 0$

Thank you for your attention

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