Stability of Finite-difference Discretizations of Singular Perturbation Problems

Relja Vulanović

Department of Mathematical Sciences
Kent State University at Stark

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Overview

- survey of techniques for proving stability for finite-difference discretizations of

$$-\varepsilon u'' - b(x)u' + c(x)u = 0, \quad x \in X := [0,1]$$

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$$0 < \varepsilon \leq \varepsilon^* \ll 1$$

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- convection-diffusion (CD):

$$b^* \geq b(x) \geq b_* > 0, \quad c^* \geq c(x) \geq 0, \quad x \in X$$

Solution properties

- RD

$$|u^{(k)}(x)| \leq M\varepsilon^{-k/2} e^{-x\sqrt{m/\varepsilon}}, \quad x \in X, \quad k = 0, 1, \ldots$$

$$c_* > m > 0$$

One layer assumed for simplicity.

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$$c_* > m > 0$$

  One layer assumed for simplicity.

- CD

$$|u^{(k)}(x)| \le M\left(1 + \varepsilon^{-k} e^{-b_* x/\varepsilon}\right), \quad x \in X, \quad k = 0, 1, \ldots$$

Special meshes

- Bakhvalov mesh B_ℓ^N, $\ell = 1$ for CD and $\ell = 2$ for RD

$$x_i = \lambda\left(\frac{i}{N}\right), \quad i = 0, 1, 2, \ldots, N$$

$$\lambda(t) = \begin{cases} \varphi(t) := \varepsilon^{1/\ell} at/(q-t) & \text{if } 0 \le t \le \tau \\ \psi(t) := \varphi'(\alpha)(t-\alpha) + \varphi(\alpha) & \text{if } \tau \le t \le 1 \end{cases}$$

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- Shishkin mesh S_ℓ^N – piecewise uniform

$$x_i = hi, \quad i = 0, 1, \ldots, J, \quad h = \frac{\eta}{J} = \frac{a\varepsilon^{1/\ell} \ln N}{J}, \quad a > 0$$

$$x_i = \eta + H(i - J), \quad i = J+1, J+2, \ldots, N, \quad H = \frac{1-\eta}{N-J}$$

RD – central scheme

- in all discrete problems: $w_0^N = U_0$, $w_N^N = U_1$

$$L_2 w_i^N := -\varepsilon D_2'' w_i^N + c_i w_i^N = 0, \quad i = 1, 2, \ldots, N-1$$

$$D_2'' w_i^N = \frac{1}{\hbar_i} \left(\frac{w_{i+1}^N - w_i^N}{h_{i+1}} - \frac{w_i^N - w_{i-1}^N}{h_i} \right)$$

$$h_i = x_i - x_{i-1}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2}$$

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- This scheme is stable uniformly in ε, i.e. the matrix A of the discrete problem satisfies

$$\|A^{-1}\| \leq M$$

in the norm induced by the maximum vector norm.

Principle 1

- Varah (1975). *Assume A is strictly diagonally dominant by rows and set* $\alpha = \min_i \left(|a_{ii}| - \sum_{j \neq i} |a_{ij}| \right)$, $\alpha > 0$. *Then* $\|A^{-1}\| < 1/\alpha$.

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- L_2 is stable uniformly in ε.
  $A = [a_{ij}]$ is tridiagonal with

$$a_{i,i-1} = -\frac{\varepsilon}{h_i \hbar_i}, \quad a_{i,i+1} = -\frac{\varepsilon}{h_{i+1}\hbar_i}$$

$$a_{ii} = -a_{i,i-1} - a_{i,i+1} + c_i, \quad i = 1, 2, \ldots, N-1.$$

Therefore,

$$|a_{ii}| - |a_{i,i-1}| - |a_{i,i+1}| = c_i, \quad i = 1, 2, \ldots, N-1,$$

and $\alpha \geq \min\{1, c_*\} > 0$.

Principle 2

- A is an L-matrix if $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$. A is inverse monotone (i.m.) if $A^{-1} \geq 0$. A is an M-matrix if it is an i.m. L-matrix.

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- Bohl (1974). *Let A be an L-matrix and let there exist a vector v^N such that $v^N > 0$ and $(Av^N)_i \geq \beta > 0$, $i = 0, 1, \ldots, N$. A is then an M-matrix and it holds that $\|A^{-1}\| \leq \beta^{-1}\|v^N\|$.*

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- For L_2, use $v^N = e^N = [1, 1, \ldots, 1]^T$. $\beta = \min\{1, c_*\}$.

RD – A hybrid 4th-order scheme

- Herceg (1990) – Hermite scheme generalized on nonuniform mesh:

$$L_4 w_i^N := -\varepsilon D_2'' w_i^N + D(cw^N)_i = 0$$
$$Dw_i^N = \gamma_i^- w_{i-1}^N + \gamma_i w_i^N + \gamma_i^+ w_{i+1}^N$$
$$\gamma_i^- = \frac{h_i^2 - h_{i+1}^2 + h_i h_{i+1}}{12 h_i \hbar_i}, \quad \gamma_i^+ = \frac{h_{i+1}^2 - h_i^2 + h_i h_{i+1}}{12 h_{i+1} \hbar_i}$$
$$\gamma_i = 1 - \gamma_i^- - \gamma_i^+ = \frac{h_i^2 + h_{i+1}^2 + 3 h_i h_{i+1}}{6 h_i h_{i+1}}$$

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- Vulanović (1993)

$$T_4 w_i^N = \begin{cases} L_4 w_i^N & \text{if } \gamma_i^- \geq 0,\ \gamma_i^+ \geq 0, \text{ and } \nu_i \leq 1 \\ L_2 w_i^N & \text{otherwise} \end{cases}$$

$$\nu_i = \frac{[(h_{i+1} + h_i)|h_{i+1} - h_i| + h_i h_{i+1}]c^*}{12\varepsilon}$$

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- This scheme is stable uniformly in ε by Principle 2.

RD – Another 4th-order scheme

- Vulanović (1997) – simpler Hermite scheme:

$$\tilde{L}_4 w_i^N := -\varepsilon D_2'' w_i^N + \tilde{D}(cw^N)_i = 0$$
$$\tilde{D} w_i^N = \tilde{\gamma}_i^- w_{i-1}^N + \tfrac{5}{6} w_i^N + \tilde{\gamma}_i^+ w_{i+1}^N$$
$$\tilde{\gamma}_i^- = \tfrac{2h_i - h_{i+1}}{12\hbar_i}, \quad \tilde{\gamma}_i^+ = \tfrac{2h_{i+1} - h_i}{12\hbar_i}$$

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- This scheme is stable uniformly in ε by Principle 1:

$$\begin{aligned} |a_{ii}| - |a_{i,i-1}| - |a_{i,i+1}| &\geq \left(\frac{5}{6} - \frac{3h_i + 3h_{i+1}}{12\hbar_i}\right) c_i + O(\hbar_i) \\ &= \frac{1}{3} c_i + \delta_i \geq \frac{1}{3} c_* + O(\hbar_i) \end{aligned}$$

RD – A 6th-order scheme

Vulanović (2004), S_2^N mesh only, $\varepsilon \le MN^{-2}$

$$T_{6,2} w_i^N := \begin{cases} L_2 w_i^N & \text{for } i = 1 \\ L_6 w_i^N & \text{for } 2 \le i \le J-2 \\ L_2 w_i^N & \text{for } J-1 \le i \le N-1 \end{cases}$$

$$\begin{aligned} L_6 w_i^N \quad := \quad & -\varepsilon D_4 w_i^N + \frac{1}{90}\Big[-(cw^N)_{i-2} + 4(cw^N)_{i-1} \\ & + 84(cw^N)_i + 4(cw^N)_{i+1} - (cw^N)_{i+2}\Big] = 0 \end{aligned}$$

$$D_4'' w_i^N = \frac{1}{12h^2}(-w_{i-2}^N + 16w_{i-1}^N - 30w_i^N + 16w_{i+1}^N - w_{i+2}^N)$$

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- Main condition

$$\begin{bmatrix} a_{k-1,k} < 0 & a_{k-1,k+1} > 0 \\ a_{kk} > 0 & a_{k,k+1} < 0 \end{bmatrix}$$

$$4a_{kk}a_{k-1,k+1} \leq a_{k-1,k}a_{k,k+1}$$

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- Some elements $a_{i,i\pm1}$ positive.
- Store the terms which cause this in matrix K, $\|K\| \le c^*/6$.

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- $B = A - K$ satisfies $\|B^{-1}\| \le 6/(5c_*)$ by Principle 3.

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- $B = A - K$ satisfies $\|B^{-1}\| \le 6/(5c_*)$ by Principle 3.
- $\|B^{-1}K\| < 1$

Principle 4

- If $A = B + K$, $\|B^{-1}\| \leq C_1$, $\|K\| \leq C_2$, $C_1 C_2 < 1$, then

$$\|A^{-1}\| = \|(I + B^{-1}K)^{-1}B^{-1}\| \leq \frac{\|B^{-1}\|}{1 - \|B^{-1}K\|} \leq \frac{C_1}{1 - C_1 C_2}.$$

Principle 4

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- $T_{6,4}$ is stable by Principle 4, Vulanović (2004).

CD – Overview

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- Principle 4 used for a two-parameter problem and a 3rd-order scheme.

CD – Upwind & central schemes

- Upwind – stable by Principle 2

$$\Lambda_1 w_i^N := -\varepsilon D_2'' w_i^N - b_i \frac{w_{i+1}^N - w_i^N}{h_{i+1}} + c_i w_i^N = 0$$

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- Central – not stable

$$\Lambda_2 w_i^N := -\varepsilon D_2'' w_i^N - b(x_i) \frac{w_{i+1}^N - w_{i-1}^N}{h_i + h_{i+1}} + c_i w_i^N = 0$$

CD – A hybrid 2nd-order scheme

- $\rho_i = \frac{b^* h_i}{2\varepsilon}$

$$P_2 w_i^N = \begin{cases} \Lambda_2 w_i^N & \text{if } \rho_i \le 1 \\ \tilde{\Lambda}_1 w_i^N & \text{otherwise} \end{cases}$$

$$\begin{aligned} \tilde{\Lambda}_1 w_i^N \quad := \quad & -\varepsilon D_2'' w_i^N - b_{i+1/2} \frac{w_{i+1}^N - w_i^N}{h_{i+1}} \\ & + c(x_{i+1/2}) \frac{w_i^N + w_{i+1}^N}{2} = 0 \end{aligned}$$

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- Stable by Principle 2.

CD – A new 3rd-order scheme

- on S_1^N only; $\chi = h$ or $\chi = H$

$$D^{(2)}_{\chi,s} w_i^N = \frac{1}{\chi^2}[(1-s)w_{i-1}^N + (3s-2)w_i^N + (1-3s)w_{i+1}^N + s w_{i+2}^N]$$

$$\begin{aligned} D^{(1)}_{\chi,s} w_i^N &= \frac{1}{6\chi}[(-3s^2+6s-2)w_{i-1}^N + 3(3s^2-4s-1)w_i^N \\ &\quad +3(-3s^2+2s+2)w_{i+1}^N + (3s^2-1)w_{i+2}^N] \end{aligned}$$

$$D^{(0)}_{\chi,s} w_i^N = [s(s-1)w_{i-1}^N + 2(1-s^2)w_i^N + s(s+1)w_{i+1}^N]/2$$

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- $\Lambda_{\chi,s} w_i := -\varepsilon D^{(2)}_{\chi,s} w_i^N - b(x_{i+s}) D^{(1)}_{\chi,s} w_i^N + c(x_{i+s}) D^{(0)}_{\chi,s} w_i^N = 0$

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- $\Lambda_{\chi,s} w_i := -\varepsilon D^{(2)}_{\chi,s} w_i^N - b(x_{i+s}) D^{(1)}_{\chi,s} w_i^N + c(x_{i+s}) D^{(0)}_{\chi,s} w_i^N = 0$
- $s = \sigma := (3-\sqrt{15})/6$ or $s = \theta := 1/\sqrt{3}$

(cont'd)

- Hybrid scheme

$$P_3 w_i^N = \begin{cases} \Lambda_{h,\sigma} w_i^N & \text{for } 1 \le i \le J-2 \\ \tilde{\Lambda}_1 w_i^N & \text{for } i = J-1, J \\ \Lambda_{H,\theta} w_i^N & \text{for } J+1 \le i \le N-2 \\ \tilde{\Lambda}_1 w_i^N & \text{for } i = N-1 \end{cases}$$

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- Hybrid scheme

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- Stable by Principle 3 if $\varepsilon^* \le M^*/N$.

Two-parameter problem

- $\mu = \varepsilon^p, \quad p > \frac{1}{2}, \quad c^* \geq c(x) \geq c_* > 0, \quad x \in X$

$$-\varepsilon u'' - \mu b(x) u' + c(x) u = 0, \quad u(0) = U_0, \quad u(1) = 0$$

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- Solution behaves like in RD; S_2^N used.
- Hybrid 3rd-order scheme, Vulanović (2001)

$$\tilde{P}_3 w_i^N = \begin{cases} \Lambda_{h,\sigma} w_i^N & \text{for } 1 \leq i \leq J-2 \\ \Lambda_2 w_i^N & \text{for } i = J-1, J, \ldots, N-1 \end{cases}$$

Two-parameter problem

- $\mu = \varepsilon^p, \quad p > \frac{1}{2}, \quad c^* \geq c(x) \geq c_* > 0, \quad x \in X$

$$-\varepsilon u'' - \mu b(x) u' + c(x) u = 0, \quad u(0) = U_0, \quad u(1) = 0$$

- Solution behaves like in RD; S_2^N used.
- Hybrid 3rd-order scheme, Vulanović (2001)

$$\tilde{P}_3 w_i^N = \begin{cases} \Lambda_{h,\sigma} w_i^N & \text{for } 1 \leq i \leq J-2 \\ \Lambda_2 w_i^N & \text{for } i = J-1, J, \ldots, N-1 \end{cases}$$

- Stable by Principle 4 if $(\varepsilon^*)^p \leq M_0/N$.