

Finitely centrally generated C^* -algebras

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Definition

A C^* -algebra is a Banach $*$ -algebra A which satisfies the C^* -identity

$$\|a^*a\| = \|a\|^2, \quad \forall a \in A.$$

Example

- (i) Let \mathcal{H} be a Hilbert space. The operator algebra $B(\mathcal{H})$ of all bounded linear operators on \mathcal{H} with the operator norm and usual adjoint obeys the C^* -identity. If \mathcal{H} is n -dimensional, we obtain that the $n \times n$ matrices $M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$ form a C^* -algebra.
- (ii) Let X be a locally compact Hausdorff space. The space $C_0(X)$ of complex-valued continuous functions on X that vanish at infinity form a commutative C^* -algebra $C_0(X)$ under pointwise operations, complex conjugation and supremum norm. $C_0(X)$ has a unit if and only if X is compact; in this case we usually write $C(X)$. More generally, if A is a C^* -algebra, then the set $C_0(X, A)$ of norm-continuous functions from X to A vanishing at infinity, with pointwise operations and supremum norm, is a C^* -algebra. In particular, $C_0(X, M_n(\mathbb{C})) \cong M_n(C_0(X)) \cong C_0(X) \otimes M_n(\mathbb{C})$ is C^* -algebra.

The morphisms in the category of C^* -algebras are **-homomorphisms*, that is, linear multiplicative maps which preserves adjoint. It is well known that every **-homomorphism* $\phi : A \rightarrow B$ between C^* -algebras A and B is contractive (hence bounded), and that ϕ is isometric if and only if ϕ is injective.

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- *faithful* if π is injective;
- *irreducible* if there is no closed invariant subspace apart from $\{0\}$ and \mathcal{H} .

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Two representations $\pi : A \rightarrow B(\mathcal{H})$ and $\rho : A \rightarrow B(\mathcal{K})$ are (*unitarily equivalent*) if there exists a unitary isomorphism $U : \mathcal{K} \rightarrow \mathcal{H}$ such that

$$\pi(a) = U\rho(a)U^*, \quad \forall a \in A.$$

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Theorem (The first Gelfand-Naimark theorem)

Let A be a commutative C^ -algebra. Then there exists a locally compact Hausdorff space X such that $A \cong C_0(X)$.*

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Theorem (The second Gelfand-Naimark theorem)

Let A be a C^ -algebra. Then there exists a Hilbert space \mathcal{H} and a faithful representation $\pi : A \rightarrow B(\mathcal{H})$.*

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Example

More generally, if E is a locally trivial C^* -bundle over the LCH base space X with fibres $M_n(\mathbb{C})$ (E is just a usual vector bundle such that the local trivializations, restricted to fibers, are isomorphisms of C^* -algebras) then the C^* -algebra $\Gamma_0(E)$ of all continuous sections vanishing at ∞ of E is n -homogeneous. In the previous example the underlying C^* -bundle E is trivial, that is $E = X \times M_n(\mathbb{C})$ (with the product topology).

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Theorem (J.M.G. Fell, Acta Math., 1961)

Let A be a n -homogeneous C^ -algebra. Then there exists a locally trivial C^* -bundle E over the locally compact Hausdorff space X whose fibres are isomorphic to $M_n(\mathbb{C})$ such that $A \cong \Gamma_0(E)$. In this case all irreducible representations of A are (up to a unitary equivalence) evaluations of sections of E at points of X .*

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If the base space X of this bundle E admits a finite open covering (U_i) such that each $E|_{U_i}$ is trivial (as a C^* -bundle), then E is said to be of *finite type* (and we shall say that in this case A is of finite type).

Each $M_n(\mathbb{C})$ -bundle E is also an n^2 -dimensional complex vector bundle (by forgetting the additional structure). If E is of finite type (as a C^* -bundle) then of course E is of finite type as a vector bundle. It is interesting (and also non-trivial) that the converse also holds. Moreover, we have the following result:

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Theorem (N.C. Phillips, TAMS, 2007)

Let X be a locally compact Hausdorff space and let E be a locally trivial $M_n(\mathbb{C})$ -bundle over X . Then the following conditions are equivalent:

- (i) E is of finite type as a C^* -bundle;*
- (ii) E is of finite type when regarded as a complex vector bundle over X by forgetting the structure;*
- (iii) E can be extended to a locally trivial $M_n(\mathbb{C})$ -bundle F over the Stone-Ćech compactification βX of X .*

Hence, to show that an $M_n(\mathbb{C})$ -bundle E is of finite type as a C^* -bundle, it is sufficient to check that the underlying n^2 -dimensional vector bundle is of finite type. The next standard fact gives a useful way to do this:

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Lemma

Let E be a locally trivial vector bundle of constant (finite) rank over a paracompact Hausdorff space X . The following conditions are equivalent:

- (i) *E is of finite type;*
- (ii) *There exists a finite number a_1, \dots, a_m of continuous bounded sections of E such that*

$$\text{span}\{a_1(x), \dots, a_m(x)\} = E(x), \quad \forall x \in X.$$

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Let A be a C^* -algebra. If A is non-unital, then there are several ways of embedding A in a unital C^* -algebra. The *multiplier algebra* of A , denoted by $M(A)$, is a unital C^* -algebra which is the largest unital C^* -algebra that contains A as an ideal in a "non-degenerate" way. It is the noncommutative generalization of Stone-Čech compactification. Of course, if A is unital then $M(A) = A$.

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$$Z(A) := \{z \in A : zx = xz, \forall x \in A\}.$$

We consider A as a $Z(M(A))$ -module, under the standard action

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Definition

A C^ -algebra A is said to be finitely centrally generated (shorter FCG) if A as a $Z(M(A))$ -module is finitely generated.*

Example

Let X be a CH space. Then the C^* -algebra $A := C(X, M_n(\mathbb{C}))$ is FCG. Indeed, since X is compact A is unital, hence $M(A) = A$. Let $(E_{i,j})$ be the standard matrix units of $M_n(\mathbb{C})$ considered as constant elements of A . Since the center of $M_n(\mathbb{C})$ consists only of the scalar multiples of identity, we have (by continuity)

$$Z(A) = \{f1_n : f \in C(X)\} \cong C(X).$$

Then for each $a = (a_{i,j}) \in A \cong M_n(C(X))$ we have $a = \sum_{i,j=1}^n (a_{i,j}1_n)E_{i,j}$, hence

$$A = \text{span}_{Z(A)}\{E_{i,j} : 1 \leq i, j \leq n\}.$$

Example

More generally, if E is a locally trivial $M_n(\mathbb{C})$ -bundle over a CH base space X then the (n -homogeneous) C^ -algebra $\Gamma(E)$ is FCG. This can be seen by using the previous example together with the finite partition of unity argument.*

Example

More generally, if E is a locally trivial $M_n(\mathbb{C})$ -bundle over a CH base space X then the (n -homogeneous) C^ -algebra $\Gamma(E)$ is FCG. This can be seen by using the previous example together with the finite partition of unity argument.*

Hence, by Fell's theorem, each unital homogeneous C^* -algebra is FCG. Of course, the same conclusion holds for a finite direct sum of unital homogeneous C^* -algebras. The converse is also true:

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Hence, by Fell's theorem, each unital homogeneous C^* -algebra is FCG. Of course, the same conclusion holds for a finite direct sum of unital homogeneous C^* -algebras. The converse is also true:

Theorem (I. Gogić, PEMS, to appear)

Let A be a C^ -algebra. Then A is finitely centrally generated if and only if A is a finite direct sum of unital homogeneous C^* -algebras.*

Sketch of the proof

Suppose that A is FCG. The proof of the theorem is divided in several steps:

- Using the functional calculus we first show that A must be unital. The easy consequence of this fact is that if A is FCG so is A/J for each (closed two-sided) ideal J of A .

Sketch of the proof

Suppose that A is FCG. The proof of the theorem is divided in several steps:

- Using the functional calculus we first show that A must be unital. The easy consequence of this fact is that if A is FCG so is A/J for each (closed two-sided) ideal J of A .
- Next, we show that A is subhomogeneous, that is the dimensions of irreducible representations of A are uniformly bounded by some finite constant. This is easy, suppose that

$$A = \text{span}_{Z(M(A))} \{e_1, \dots, e_m\}$$

for some $e_1, \dots, e_m \in A$. If π is an irreducible representation of A then π can be extended (in a unique way) to the irreducible representation $\tilde{\pi}$ of $M(A)$ (on the same Hilbert space). Then $\tilde{\pi}$ maps $Z(M(A))$ into scalars, so

$$\pi(A) = \text{span}_{\mathbb{C}} \{\pi(e_1), \dots, \pi(e_m)\} \Rightarrow \dim \pi \leq \sqrt{m} < \infty.$$

- Suppose that A is subhomogeneous of degree n (i.e. the maximal dimension of irreducible representation of A equals n) and let J be the n -homogeneous ideal of A (J is the intersection of the kernels of all irreducible representations of dimension at most $n - 1$). To prove that A is a finite direct sum of unital homogeneous C^* -algebras, note that it is sufficient to show that J is unital. Indeed, in this case $A \cong J \oplus (A/J)$, where A/J is FCG with the lower degree of subhomogeneity.

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- Now, we show that J is of finite type. To see this, let E be a locally trivial $M_n(\mathbb{C})$ -bundle over the LCH base space X such that $J \cong \Gamma_0(E)$. Using the previous lemma, we see that E must be of finite type as a vector bundle, and hence, by Phillips's theorem, E is of finite type as a C^* -bundle.

- Next, we reduce the proof to the case when J is essential in A (i.e. if I is any ideal of A such that $IJ = \{0\}$ then $I = \{0\}$). In this case, $A \subseteq M(J)$, and by [3] we have the equalities

$$M(J) = \Gamma_b(E) = \Gamma(F),$$





where $\Gamma_b(E)$ denotes the C^* -algebra of all continuous bounded sections of E and F denotes the $M_n(\mathbb{C})$ -bundle over βX which extends E (such F exists by Phillips's theorem).

- Finally, to obtain a contradiction, we assume that J is non-unital so that X is non-compact. In this case it can be seen that there exists a point $s_0 \in \beta X \setminus X$, a compact neighborhood H of s_0 and an ideal I_H of $M(J)$ (which consists of all $a \in M(J)$ such that $a|_H = 0$) such that $A_H := A/(I_H \cap A)$ can be identified with a C^* -subalgebra of $C(H, M_n(\mathbb{C}))$ and

$$a_{1,n}|_{H \setminus U} = 0, \quad \forall a = (a_{i,j})_{1 \leq i,j \leq n} \in A_H,$$

where $U := X \cap H$. Note that U is a dense open subset of H , and $s_0 \notin U$. Using this fact we then show that the commutative C^* -algebra $C_0(U)$ is FCG. By the first part of the proof we conclude that $C_0(U)$ must be unital, so that U is compact, hence equal to H , contradicting the fact that $s_0 \in H \setminus U$.

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