The Gram matrix in inner product modules over C^*-algebras

Ljiljana Arambašić

(joint work with D. Bakić and M.S. Moslehian)

Department of Mathematics
University of Zagreb

Applied Linear Algebra
May 24–28, Novi Sad

Definition

A C^*-**algebra** is a Banach $*$-algebra $\mathcal{A}$ such that $\|a^*a\| = \|a\|^2, \forall a \in \mathcal{A}$.
A right $\mathcal{A}$-module $\mathcal{X}$ is a **semi-inner product $\mathcal{A}$-module** if there is an $\mathcal{A}$-semi-inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$, i.e. the mapping satisfying

1. $\langle x, x \rangle \geq 0$;
2. $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$;
3. $\langle x, ya \rangle = \langle x, y \rangle a$;
4. $\langle x, y \rangle^* = \langle y, x \rangle$.

for all $x, y, z \in \mathcal{X}, \lambda \in \mathbb{C}, a \in \mathcal{A}$.

Let $\mathcal{X}$ be a semi-inner product $\mathcal{A}$-module. We define a semi-norm on $\mathcal{X}$ by

$$\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}, \quad x \in \mathcal{X}$$

where the latter norm denotes that in the C^*-algebra $\mathcal{A}$.

Examples of C^*-algebras:

1. $C([0,1]) = \{f : [0,1] \to \mathbb{C} : \text{f is continuous}\}$ with $\|f\| = \max\{|f(t)| : t \in [0,1]\}$.
2. $\mathbb{B}(H)$ with the operator norm and the usual adjoint.
3. In particular, if H is n-dimensional, then $M_n(\mathbb{C}) \cong \mathbb{B}(H)$ is a C^*-algebra.
4. Every closed $*$-subalgebra of $\mathbb{B}(H)$ is a C^*-algebra.

Examples of semi-inner product modules:

1. Every semi-inner product space is a semi-inner product $\mathbb{C}$-module.
2. Each C^*-algebra $\mathcal{A}$ can be regarded as a semi inner product $\mathcal{A}$-module via $\langle a, b\rangle = a^*b$ $(a, b \in \mathcal{A})$.
3. For every pair of Hilbert spaces H_1 and H_2, the space $\mathbb{B}(H_1, H_2)$ of all bounded linear operators from H_1 to H_2 is a Hilbert $\mathbb{B}(H_1)$-module with the inner product $\langle T, S\rangle = T^*S$.

Definition

Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be a semi-inner product $\mathcal{A}$-module and $n \in \mathbb{N}$. The Gram matrix of elements $x_1, \ldots, x_n \in \mathcal{X}$ is defined as the matrix

$$[\langle x_i, x_j \rangle] \in M_n(\mathcal{A}).$$

- For every $x_1, \ldots, x_n \in \mathcal{X}$ it holds $[\langle x_i, x_j \rangle] \geq 0$ in $M_n(\mathcal{A})$.
- The Cauchy-Schwarz inequality for $x, y \in \mathcal{X}$

$$\langle x, y \rangle \langle y, x \rangle \leq \|y\|^2 \langle x, x \rangle \Leftrightarrow \begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle x, y \rangle^* & \|\langle y, y \rangle\| e \end{bmatrix} \geq 0.$$

Positivity of the Gram matrix sharpens the Cauchy-Schwarz inequality, since

$$\begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle x, y \rangle^* & \|\langle y, y \rangle\| e \end{bmatrix} \geq \begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle x, y \rangle^* & \langle y, y \rangle \end{bmatrix} \geq 0.$$

Let $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ be a semi-inner product $\mathcal{A}$-module. For $z \in \mathcal{X}, \langle z, z\rangle \neq 0$, we define

$$\langle\cdot,\cdot\rangle_z : \mathcal{X} \times \mathcal{X} \to \mathcal{A}, \quad \langle x, y\rangle_z := \|z\|^2 \langle x, y\rangle - \langle x, z\rangle\langle z, y\rangle.$$

Observe that $\langle x, x\rangle_z = \|z\|^2 \langle x, x\rangle - \langle x, z\rangle\langle z, x\rangle \geq 0$. Then $\langle\cdot,\cdot\rangle_z$ is another semi-inner product on $\mathcal{X}$ and therefore $\left[\langle x_i, x_j\rangle_z\right] \geq 0$, that is,

$$\left[\,\langle x_i, x_j\rangle\,\right] \geq \frac{1}{\|z\|^2}\left[\,\langle x_i, z\rangle\,\langle z, x_j\rangle\,\right] = \left[\,\langle x_i, z\rangle \left(\frac{1}{\|z\|^2} e\right) \langle z, x_j\rangle\,\right].$$

Replacing $\langle\cdot,\cdot\rangle$ with $\langle\cdot,\cdot\rangle_z$ in the last inequality we get

$$\left[\,\langle x_i, x_j\rangle\,\right] \geq \left[\,\langle x_i, z\rangle \left(\frac{1}{\|z\|^2} e + b\right) \langle z, x_j\rangle\,\right]$$

where $b := \frac{1}{\|z\|^2 \cdot \| \|z\|^2 \langle z,z\rangle - \langle z,z\rangle^2 \|}(\|z\|^2 e - \langle z, z\rangle)^2 \in \mathcal{A}^+$. We proceed by induction.

Theorem

Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be a semi-inner product $\mathcal{A}$-module, $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{X}$. Let $z \in \mathcal{X}$, $\langle z, z \rangle \neq 0$. Then there is a non-decreasing sequence $(p_m(\langle z, z \rangle))_m \in \mathcal{A}^+$ such that

$$\begin{aligned}[\langle x_i, x_j \rangle] &\geq \ldots \geq [\langle x_i, z \rangle\, p_m(\langle z, z \rangle)\, \langle z, x_j \rangle] \geq [\langle x_i, z \rangle\, p_{m-1}(\langle z, z \rangle)\, \langle z, x_j \rangle] \\ &\geq \ldots \geq [\langle x_i, z \rangle\, p_0(\langle z, z \rangle)\, \langle z, x_j \rangle] = \frac{1}{\|z\|^2}[\langle x_i, z \rangle\, \langle z, x_j \rangle] \geq 0.\end{aligned}$$

Theorem

Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be a semi-inner product $\mathcal{A}$-module, $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{X}$. Let $z \in \mathcal{X}$, $\langle z, z \rangle \neq 0$. If $b \in \mathcal{A}^+$ is such that $\|zb^{\frac{1}{2}}\| \leq 1$, then

$$\left[\, \langle x_i, x_j \rangle \,\right] \geq \left[\, \langle x_i, z \rangle\, b\, \langle z, x_j \rangle \,\right].$$

It holds that $\|zp_m(\langle z, z \rangle)^{\frac{1}{2}}\| = 1$, $m \in \mathbb{N}$.

Let $\mathcal{A}$ be a unital C^*-algebra. For a positive element $a \in \mathcal{A}$, $a \neq 0$, define

$$\begin{aligned} f_0(a) &= a, & g_0(a) &= \|f_0(a)\|e - f_0(a), \\ f_m(a) &= f_{m-1}(a)g_{m-1}(a), & g_m(a) &= \|f_m(a)\|e - f_m(a), \quad m \in \mathbb{N}. \end{aligned}$$

For m such that $f_m(a) \neq 0$ we define

$$\begin{aligned} p_0(a) &= \tfrac{e}{\|f_0(a)\|}, \\ p_1(a) &= \tfrac{e}{\|f_0(a)\|} + \tfrac{g_0(a)^2}{\|f_0(a)\|\cdot\|f_1(a)\|}, \\ p_2(a) &= \tfrac{e}{\|f_0(a)\|} + \tfrac{g_0(a)^2}{\|f_0(a)\|\cdot\|f_1(a)\|} + \tfrac{g_0(a)^2 g_1(a)^2}{\|f_0(a)\|\cdot\|f_1(a)\|\cdot\|f_2(a)\|}, \\ &\dots \\ p_m(a) &= \tfrac{e}{\|f_0(a)\|} + \sum_{l=1}^{m} \left(\tfrac{1}{\prod_{k=0}^{l} \|f_k(a)\|} \prod_{k=0}^{l-1} g_k(a)^2 \right). \end{aligned}$$

If $f_m(a) \neq 0$ and $f_{m+1}(a) = 0$ then we define

$$p_j(a) = p_m(a), \quad \forall j > m.$$

Suppose $\mathcal{A} \subseteq \mathbb{B}(H)$.

- $f_m(a), m \in \mathbb{N}$ are polynomials in a:

$$\begin{aligned} f_0(\lambda) &= \lambda, \\ f_1(\lambda) &= \|a\|\lambda - \lambda^2, \\ f_m(\lambda) &= \|f_{m-1}(a)\| f_{m-1}(\lambda) - f_{m-1}(\lambda)^2, \quad m \in \mathbb{N}. \end{aligned}$$

  Observe that $\deg f_m = 2^m$ for all m.
- $f_m(a) \geq 0$ for all $m \in \mathbb{N}$.
- $\|f_{m+1}(a)\| \leq \frac{1}{4}\|f_m(a)\|^2$.
- Suppose that there is $m \in \mathbb{N}$ such that $f_m(a) = 0$. Let $\lambda \in \sigma(a)$. Then

$$f_m(\lambda) \in f_m(\sigma(a)) = \sigma(f_m(a)) = \{0\}.$$

  This shows that $\sigma(a)$ is contained in a finite set, namely in the set of all zeros of the polynomial f_m.

- Suppose $\sigma(a)$ is a finite set. Let $\sigma(a) \setminus \{0\} = \{\lambda_1, \lambda_2, \dots \lambda_k\}$ where $\lambda_1 > \lambda_2 > \dots > \lambda_k > 0$. By the spectral theory there exist orthogonal projections $P_1, P_2, \dots P_k \in \mathbb{B}(H)$ which are mutually orthogonal and such that $a = \sum_{i=1}^{k} \lambda_i P_i$.

$$f_0(a) = a = \sum_{i=1}^{k} \lambda_i P_i, \quad \|f_0(a)\| = \lambda_1.$$

$$f_1(a) = \lambda_1 f_0(a) - f_0(a)^2 = \sum_{i=2}^{k} \lambda_i(\lambda_1 - \lambda_i) P_i.$$

This shows that $f_1(a)$ has at most $k-1$ non-zero elements in its spectrum. Suppose that $\lambda_2(\lambda_1 - \lambda_2) \geq \lambda_i(\lambda_1 - \lambda_i)$ for all $i = 2, \dots k$. Then $\|f_1(a)\| = \lambda_2(\lambda_1 - \lambda_2)$ and

$$f_2(a) = \sum_{i=3}^{k} (\lambda_1 - \lambda_i)(\lambda_2 - \lambda_i)(\lambda_1 - \lambda_2 - \lambda_i)\lambda_i P_i.$$

- Let a be a positive element of a C^*-algebra $\mathcal{A} \subseteq \mathbb{B}(H)$. Then there exists $m \in \mathbb{N}$ such that $f_m(a) = 0$ if and only if a has a finite spectrum.
- If such m exists, then $m \leq \operatorname{card} \sigma(a)$.
- If $\mathcal{A}$ is finite-dimensional, then $\sigma(a)$ is finite.
- Suppose that $\mathcal{A} = \mathbb{C}$, i.e. that $\mathcal{X}$ is a semi-inner product space. Then for each $z \in \mathcal{X}$ the spectrum $\sigma(\langle z, z\rangle)$ is a singleton, so $f_1(\langle z, z\rangle) = 0$. Hence, in this situation, we have only one inequality:

$$\left[\, \langle x_i, x_j\rangle \,\right] \geq \frac{1}{\|z\|^2}\left[\, \langle x_i, z\rangle \, \langle z, x_j\rangle \,\right] \geq 0.$$

Example

Let $a = \mathrm{diag}(5,4,2,1)$. Here we have

$$f_0(a) = \mathrm{diag}(5,4,2,1), \quad \|f_0(a)\| = 5, \quad p_0(a) = \mathrm{diag}(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}),$$

$$f_1(a) = \mathrm{diag}(0,4,6,4), \quad \|f_1(a)\| = 6, \quad p_1(a) = \mathrm{diag}(\tfrac{1}{5}, \tfrac{7}{30}, \tfrac{1}{2}, \tfrac{11}{15}),$$

$$f_2(a) = \mathrm{diag}(0,8,0,8), \quad \|f_2(a)\| = 8, \quad p_2(a) = \mathrm{diag}(\tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{2}, 1),$$

$$f_3(a) = 0 \qquad \Rightarrow \qquad p_k(a) = p_2(a), \quad k \geq 3.$$

Observe that

$$p_2(a) = a^{-1}.$$

Theorem

Let $\mathcal{A} \subseteq \mathbb{B}(H)$ be a C^*-algebra and $a \in \mathcal{A}^+$, $a \neq 0$ such that $\sigma(a)$ is finite. Let M be the number with the property $f_M(a) \neq 0$ and $f_{M+1}(a) = 0$. Then:

- *$ap_M(a)$ is the orthogonal projection to the image of a.*
- *In particular, if a is an invertible operator, then $p_M(a) = a^{-1}$.*

Sketch of the proof.

- For every $\lambda \in \sigma(a)$ there is $m \leq M$ such that $f_m(\lambda) = \|f_m(a)\|$.
- If $f_m(\lambda_m) = \|f_m(a)\|$ then $p_j(\lambda_m) = \frac{1}{\lambda_m}$ for all $j \geq m$.
- $\lambda p_M(\lambda) = \begin{cases} 1, & \lambda \in \sigma(a) \setminus \{0\}, \\ 0, & \lambda \in \sigma(a) \cap \{0\}. \end{cases}$

□

Theorem

Let a be a positive element in a C^*-algebra $\mathcal{A} \subseteq \mathbb{B}(H)$ with an infinite spectrum. Then

$$\lim_{m\to\infty} a^2 p_m(a) = a.$$

In particular, if a is an invertible operator, $\lim_{m\to\infty} p_m(a) = a^{-1}$.

Remark

From $\lim_{m\to\infty} a^2 p_m(a) = a$ one easily gets $\lim_{m\to\infty} a p_m(a) = p$ in the strong operator topology.

Example

- Suppose that a is a positive compact operator with an infinite spectrum. Then $ap_m(a)$ is compact operator for every m.
- If the sequence $(ap_m(a))$ converges in norm, then the limit has to be a compact operator.
- Let p be the orthogonal projection to $\overline{\operatorname{Im} a}$. Since $\sigma(a)$ is an infinite set, $\overline{\operatorname{Im} a}$ is an infinite dimensional subspace and hence p is a non-compact operator.
- Therefore, the sequence $(ap_m(a))$ does not converge to p in norm.

Proposition

Let $a \in \mathbb{B}(H)$ be a positive operator and $p \in \mathbb{B}(H)$ the orthogonal projection to $\overline{Im\, a}$. Then $(ap_m(a))_m$ converges to p in norm if and only if $Im\, a$ is a closed subspace of H.

At the end, let us turn back to the Gram matrix.

Proposition

Let $\mathcal{X}$ be a semi-inner product module over a C^*-algebra $\mathcal{A} \subseteq \mathbb{B}(H)$. For $z \in \mathcal{X}$ and $a = \langle z, z \rangle \in \mathcal{A}$, let $p \in \mathbb{B}(H)$ denotes the orthogonal projection to $\overline{Im\, a}$. Suppose that there exists a positive operator $h \in \mathbb{B}(H)$ such that for all $x_1, \dots, x_n \in \mathcal{X}$ and every $m \geq 0$ it holds

$$[\langle x_i, x_j \rangle] \geq [\langle x_i, z \rangle \, h \, \langle z, x_j \rangle] \geq [\langle x_i, z \rangle \, p_m(a) \, \langle z, x_j \rangle].$$

Then $aha = a$ and $ah = p$.
If $\sigma(a)$ is finite and $M \in \mathbb{N}$ such that $ap_M(a) = p$, then h and $p_M(a)$ coincide on $Im\, a$.