ON AN EFFICIENT FAMILY OF SIMULTANEOUS METHODS FOR FINDING POLYNOMIAL ZEROS

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THE AIM OF COMMUNICATION

• **The aim of this communication is to present a new family of iterative methods for the simultaneous determination of complex polynomial zeros which is ranked as the most efficient among existing methods based on fixed point relations.**

• **The presented family of methods relies on the fixed point relation of Gargantini-Henrici's type and a class of suitable corrections which enable very fast convergence (equal to six) with the minimal computational costs.**

[I. Gargantini, P. Henrici, Numer. Math. 18 (1972)]

WHY SIMULTANEOUS DETERMINATION OF ZEROS?

• The problem of solving polynomial equations is very important in the theory and practice, for example, in

– applied mathematics,

– many branches of engineering sciences (e.g., control theory, digital signal processing, nonlinear circuits, analysis of transfer functions),

– physics (high-energy physics), computer science,

– finance,

– biology, etc.

See [J. M. McNamee, Numerical Methods for Roots of Polynomials, Elsevier 2007].

WHY SIMULTANEOUS DETERMINATION OF ZEROS?

(CONTINUATION)

Determination of polynomial zeros simultaneously is useful and efficient because of

- **self-correction of root approximations during the iteration process (not possible by methods of other type),**
- **very fast convergence,**
- **low computational cost in the implementation on digital computers,**
- **very efficient parallelism since several versions of the same algorithm can run simultaneously.**

FAMILY OF ACCELERATED SIMULTANEOUS METHODS

$f(z) = \prod_{j=1}^{n}(z-\zeta_j)$ – a monic polynomial of degree n with simple, real or complex zeros $\zeta_1, \ldots, \zeta_n$

From

$$u(z) = \frac{f(z)}{f'(z)} = \Big[\frac{d}{dz}\log f(z)\Big]^{-1} = \Big(\sum_{j=1}^{n}\frac{1}{z-\zeta_j}\Big)^{-1} \tag{1}$$

the following basic relation follows:

$$\zeta_i = z - \frac{1}{\dfrac{1}{u(z)} - \displaystyle\sum_{\substack{j\in I_n \\ j\neq i}} \frac{1}{z-\zeta_j}} \quad (i \in I_n := \{1,\ldots,n\}). \tag{2}$$

Let $z_1, \ldots, z_n$ be distinct approximations to the zeros $\zeta_1, \ldots, \zeta_n$.

Setting $z = z_i$ and substituting the zeros ζ_j by some approximations z_j^* in

$$\zeta_i = z - \frac{1}{\dfrac{1}{u(z)} - \sum\limits_{\substack{j \in I_n \\ j \neq i}} \dfrac{1}{z - \zeta_j}} \quad (i \in I_n := \{1, \ldots, n\}),$$

the following iterative method

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{u(z_i)} - \sum\limits_{\substack{j \in I_n \\ j \neq i}} \dfrac{1}{z_i - z_j^*}} \quad (i \in I_n) \tag{3}$$

for the simultaneous determination of all simple zeros of the polynomial P is obtained.

The choice $z_j^* = z_j$ in (3) gives the well-known Ehrlich-Aberth method

$$\hat{z}_i = z_i - \cfrac{1}{\cfrac{1}{u(z_i)} - \sum\limits_{\substack{j \in I_n \\ j \neq i}} \cfrac{1}{z_i - z_j}} \quad (i \in I_n), \tag{4}$$

of the third order.

The choice $z_j^* = z_j - u(z_j)$ in (3) gives the Nourein method

$$\hat{z}_i = z_i - \cfrac{1}{\cfrac{1}{u(z_i)} - \sum\limits_{\substack{j \in I_n \\ j \neq i}} \cfrac{1}{z_i - z_j + u(z_j)}} \quad (i \in I_n) \tag{5}$$

of the fourth order

(NO ADDITIONAL CALCULATIONS ARE NEEDED!).

Considering the Ehrlich-Abert method (4) and the Nourein method (5), it is evident that the better approximations z_j^* in

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{u(z_i)} - \sum\limits_{\substack{j \in I_n \\ j \neq i}} \dfrac{1}{z_i - z_j^*}} \quad (i \in I_n)$$

give the more accurate approximations $\hat{z}_i$; indeed, if $z_j^* \longrightarrow \zeta_j$, then $\hat{z}_i \longrightarrow \zeta_i$.

In this paper we extend such an approach to state a family of sixth-order methods.

Let h be a real or complex function such that h and its derivatives h' and h'' are continuous in the neighborhood of 0, and let

$$u_j = u(z_j), \quad t_j = \frac{f(z_j - u_j)}{f(z_j)}.$$

Assume that approximations z_j^*, appearing in

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{u(z_i)} - \displaystyle\sum_{\substack{j \in I_n \\ j \neq i}} \frac{1}{z_i - z_j^*}} \quad (i \in I_n),$$

are given by

$$z_j^* = z_j - u_j - h(t_j)\frac{f(z_j - u_j)}{f'(z_j)}.$$

We state the NEW SIMULTANEOUS METHOD:

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{u_i} - \displaystyle\sum_{\substack{j\in I_n\\ j\neq i}} \Bigg(z_i - \underbrace{z_j + u_j + h(t_j)\frac{f(z_j - u_j)}{f'(z_j)}}_{z_j^*}\Bigg)^{-1}} \quad (i \in I_n). \tag{6}$$

The function h should be determined in such a way to provide the order of convergence of the family of methods (6) as high as possible.

CONVERGENCE THEOREM

The main result of the paper is the convergence theorem that gives necessary and sufficient conditions for the function h to provide as high as possible order of convergence of the simultaneous method (6).

THEOREM 1. Let h be any function satisfying

$$h(0) = 1,\ h'(0) = 2, \quad \text{and} \quad |h''(0)| < \infty.$$

If $z_1^{(0)}, \ldots, z_n^{(0)}$ are sufficiently close initial approximations to the distinct zeros $\zeta_1, \ldots, \zeta_n$, then the order of convergence of the family of simultaneous methods (6) is six.

The proof is based on the error relation:

$$\begin{aligned}\hat{\varepsilon}_i \;=\; & -\varepsilon_i^2\Big\{c_{2,j}\Big[1-h(0)\Big]\varepsilon_j^2 \\ & +\Big[-2c_{3,j}(h(0)-1)+c_{2,j}^2(4h(0)-h'(0)-2)\Big]\varepsilon_j^3+ \\ & \Big[-3c_{4,j}(h(0)-1)+c_{2,j}c_{3,j}(-7+14h(0)-4h'(0)) \\ & +c_{2,j}^3(4-13h(0)+7h'(0)-h''(0)/2)\Big]\varepsilon_j^4\Big\}+O_M(\varepsilon^7).\end{aligned}$$

$$c_{k,j}=\frac{f^{(k)}(\zeta_j)}{k!\,f'(\zeta_j)}$$

To provide the order six, we must take

$h(0)=1,\ h'(0)=2$ **and** $|h''(0)|$ **is bounded.**

THE CHOICE OF FUNCTION $h(t)$

Example 1.

$$h_1(t) = \frac{1+\beta t}{1+(\beta-2)t} \quad (\beta \in \mathbb{R})$$

Example 2.

$$h_2(t) = \left(1+\frac{2}{m}t\right)^m \quad (m \neq 0 \text{ is a rational number})$$

Example 3.

$$h_3(t) = \frac{1+\gamma t^2}{1-2t}, \quad (\gamma \in \mathbb{R})$$

Example 4.

$$h_4(t) = \frac{1}{1-2t+at^2} \quad (a \in \mathbb{R})$$

Example 5.

$$h_5(t) = \frac{t^2 + (c-2)t - 1}{ct - 1} \quad (c \in \mathbb{R})$$

Example 6.

$$h_6(t) = \frac{1}{t}\Big(\frac{2}{1+\sqrt{1-4t}} - 1\Big)$$

ANOTHER CONTRIBUTION: Iterative functions of the form

$$z^* \;=\; \phi(z) = z - u(z) - h(t)\frac{f(z-u(z))}{f(z)}$$
$$(h(0) = 1,\ h'(1) = 2,\ |h''(0)| < \infty),$$

including h_1 to h_6, define some new and some existing optimal two-point methods of the fourth order for finding a simple zero of a nonlinear equation.

COMPUTATIONAL ASPECTS

From a practical point of view, it is of great importance to know the computational efficiency of any iterative zero-finding method since it is closely connected to the features such as

- **the number of necessary numerical operations in computing zeros with the required accuracy,**
- **the convergence speed,**
- **processor time of a computer, etc.**

The knowledge of the computational efficiency is of particular interest in designing a package of root-solvers.

The efficiency of an iterative method (IM) can be successfully estimated using the coefficient of efficiency given by

$$E(IM) = \frac{\log r}{d}, \tag{7}$$

where

r is the order of convergence of the iterative method (IM), and

d is the computational cost.

The rank list of methods obtained by this formula mainly matches well with the real CPU (central processor unit) time, see Chapter 6 of

M. S. Petković, Iterative Methods for Simultaneous Inclusion of Polynomial Zeros, Springer-Verlag, 1989, 2008.

The computation cost d is usually calculated using arithmetic operations per iteration taken with certain **weights** depending on the number of bits b of the used computer arithmetic:

$w_{as} \sim O(b)$ – addition+subtraction,

$w_m \sim O(b \log b \log(\log b))$ – multiplication [Schönhage, Strassen]

$w_d \sim 4w_m$ – division.

The number of basic operations per one iteration for all n zeros:

AS_n – addition+subtraction

M_n – multiplication

D_n – division

Computational cost:

$$d = d(n) = w_{as}AS_n + w_m M_n + w_d D_n$$

Coefficient of efficiency:

$$E(IM, n) = \frac{\log r}{w_{as}AS_n + w_m M_n + w_d D_n}. \tag{8}$$

WE COMPARED the convergence behavior and computational efficiency of

– the Ehrlich-Aberth method (4) of order 3,

– the Nourein method (5) of order 4

– the new family of simultaneous method (6) of order 6.

Methods	$A_n + S_n$	M_n	D_n
The Ehrlich-Abert method (4)	$4n^2 - 2n$	$2n^2$	$n^2 + n$
The Nourein method (5)	$4n^2 - n$	$2n^2$	$n^2 + n$
The new method (6)	$5n^2 + n$	$3n^2 + 2n$	$n^2 + 2n$

Table 1 The number of basic operations (real arithmetic operations)

The data for the weights of arithmetic operations were taken from

R. Brent, P. Zimmermann, Modern Computer Arithmetic, Cambridge University Press (to appear), available electronically as Version 0.5.1.

Applying (8) we calculated the percent ratios

$$\rho_{6,4}(n) = (E((6),n)/E((4),n) - 1) \cdot 100 \quad (\text{in } \%), \qquad (\text{F/EA\%})$$

$$\rho_{6,5}(n) = (E((6),n)/E((5),n) - 1) \cdot 100 \quad (\text{in } \%). \qquad (\text{F/N\%})$$

These ratios are graphically presented in Figure 1 as functions of the polynomial degree n and show the (percentage) improvement of computational efficiency of the new method (6) in relation to the methods (4) and (5).

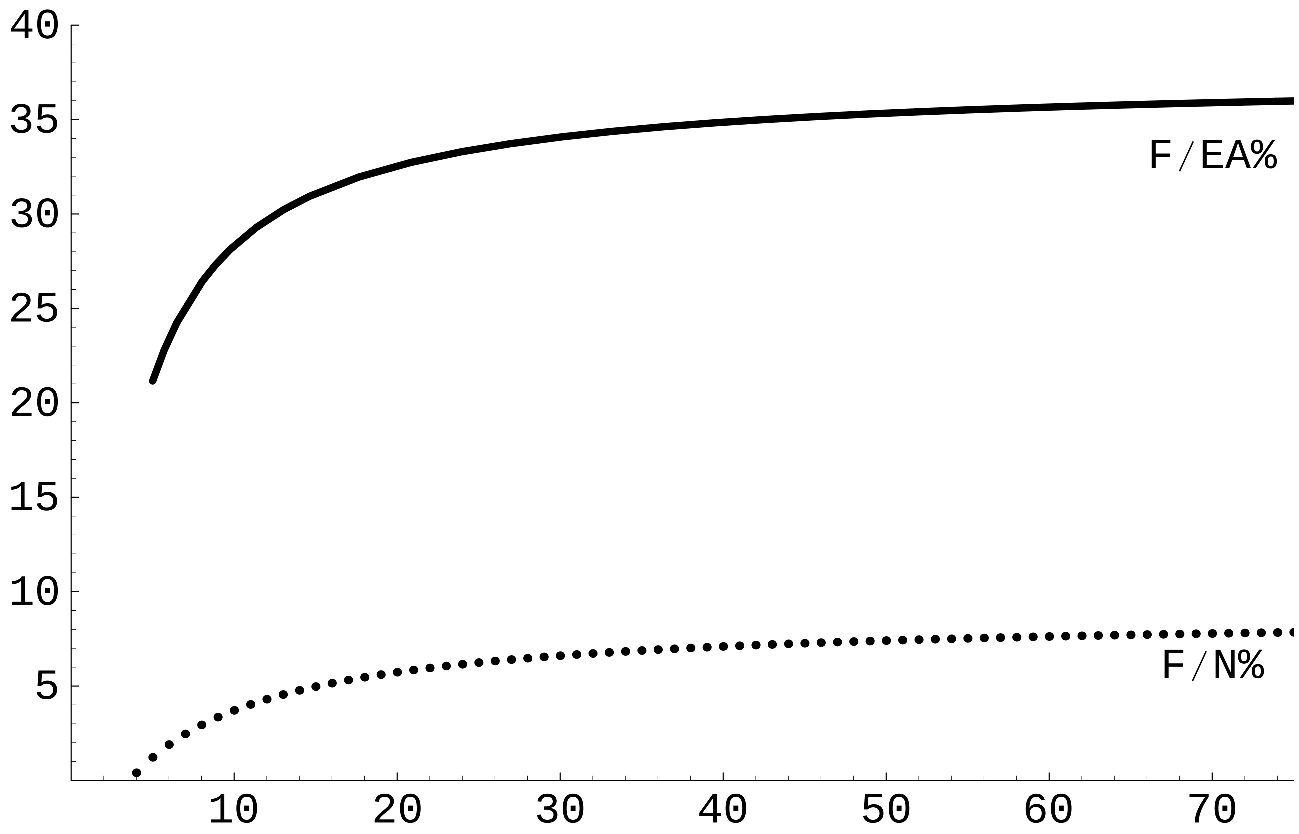

Fig. 1 Ratios of computational efficiency

It is evident from Figure 1 that the new method (F-6) is more efficient than the methods (E-4) and (N-5). The improvement is especially expressive in regard to the Ehrlich-Aberth method (E-4) (full line).

NUMERICAL EXAMPLES

To demonstrate the convergence behavior of the methods (4), (5) and (6), we tested a number of polynomial equations; for illustration, we present two examples.

We applied the programming package Mathematica with multiprecision arithmetic based on the GNU multiprecision package GMP developed by Granlund

T. Granlund, GNU MP; The GNU multiple precision arithmetic library, edition 2.0 (1996).

As a measure of accuracy of the obtained approximations, we have calculated Euclid's norm

$$e^{(m)} := ||z^{(m)} - \zeta||_2 = \left(\sum_{i=1}^{n} \left|z_i^{(m)} - \zeta_i\right|^2\right)^{1/2} \quad (m = 0, 1, \ldots).$$

The tables also contain the computational order of convergence $\tilde{r}$, evaluated by the following formula

$$\tilde{r} \approx \frac{\log |e_{k+1}/e_k|}{\log |e_k/e_{k-1}|}.$$

EXAMPLE 1.

$$P_{17}(z) = (z-1)(z^8-256)(z^8-65536)$$

The denotation $A(-h)$ means $A \times 10^{-h}$.

Methods	$e^{(1)}$	$e^{(2)}$	$e^{(3)}$	$\tilde{r}$
The Ehrlich-Aberth method (4)	$6.04(-2)$	$2.37(-5)$	$1.28(-15)$	3.0139
The Nourein method (5)	$2.69(-2)$	$4.19(-8)$	$7.31(-32)$	4.0910
(M_1)–h_1, $\beta = 0$	$5.17(-3)$	$3.98(-16)$	$3.55(-96)$	6.1047
(M_2)–h_2, $m = 2$	$4.56(-2)$	$9.82(-10)$	$3.73(-55)$	5.9245
(M_3)–h_3, $\gamma = 1$	$1.42(-2)$	$1.23(-13)$	$9.69(-80)$	5.9752
(M_4)–h_4, $a = -1$	$4.55(-3)$	$1.92(-16)$	$1.36(-95)$	5.9184
(M_5)–h_5, $c = 1$	$1.19(-2)$	$3.21(-14)$	$2.87(-83)$	5.9674
(M_6)–h_6	$1.27(-2)$	$8.27(-14)$	$1.92(-84)$	6.3147

Table 2 Euclid's norm of errors – the polynomial of the 17th degree

EXAMPLE 2.

$$P_{21}(z) = (z-4)(z^2-1)(z^4-16)(z^2+9)(z^2+16)(z^2+2z+5)\times$$
$$(z^2+2z+2)(z^2-2z+2)(z^2-4z+5)(z^2-2z+10).$$

Methods	$e^{(1)}$	$e^{(2)}$	$e^{(3)}$	$\tilde{r}$
The Ehrlich-Aberth method (4)	$8.76(-2)$	$1.03(-4)$	$2.16(-13)$	2.9622
The Nourein method (5)	$4.61(-2)$	$5.74(-7)$	$1.26(-26)$	4.0080
(M_1)–$h_1,\ \beta=0$	$1.40(-2)$	$3.14(-12)$	$4.20(-70)$	5.9978
(M_2)–$h_2,\ m=2$	$2.61(-2)$	$5.72(-10)$	$4.85(-56)$	6.0152
(M_3)–$h_3,\ \gamma=1$	$1.27(-2)$	$1.94(-12)$	$3.31(-71)$	5.9869
(M_4)–$h_4,\ a=-1$	$6.36(-3)$	$2.59(-14)$	$3.10(-83)$	6.0510
(M_5)–$h_5,\ c=1$	$2.54(-2)$	$5.06(-10)$	$2.26(-56)$	6.0189
(M_6)–h_6	$2.44(-2)$	$1.14(-11)$	$1.54(-67)$	5.9878

Table 3 Euclid's norm of errors – the polynomial of the 21-st degree

CONCLUSIONS:

• From Tables 2 and 3 and a number of tested polynomial equations we can conclude that the proposed family (6) produces approximations of exceptional accuracy; two iterative steps are usually sufficient in solving most practical problems when initial approximations are reasonably good and polynomials are well-conditioned.

• The presented analysis of computational efficiency shows that the proposed family (6) is more efficient than all existing methods based on fixed point relations.

• We observe that the computational order of convergence (the last column of Tables 2 and 3) mainly well coincides to the theoretical order of convergence of all considered methods.


THANK YOU FOR YOUR ATTENTION!