Bootstrap AMG for Markov Chain Computations

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Outline

Finite homogeneous Markov Chains

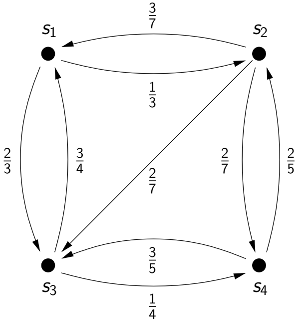
- States $s_1, \ldots, s_n, n \in \mathbb{N}$
- Transition probabilities $p(s_j \rightarrow s_i)$

$\Rightarrow$ Transition matrix $T = (t_{ij})_{i,j} \in \mathbb{R}^{n \times n}$

$$t_{ij} = p(s_j \rightarrow s_i)$$

- Example:

$$T = \begin{pmatrix} 0 & \frac{3}{7} & \frac{3}{4} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{5} \\ \frac{2}{3} & \frac{2}{7} & 0 & \frac{3}{5} \\ 0 & \frac{2}{7} & \frac{1}{4} & 0 \end{pmatrix}$$



Properties of Markov Chains

- Column-stochastic transition matrices T,

$$\mathbf{1}^t T = \mathbf{1}^t, \quad \mathbf{1}^t = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$$

- T irreducible $\Rightarrow$ existence of unique steady-state solution

$$Tx = x, \quad x > 0, \|x\|_1 = 1$$

- $\text{spec}_r(B) \subset \mathcal{U}_1(0) = \{z \in \mathbb{C}, |z| < 1\}$
- Equivalent formulation as system of linear equations

$$Ax = (I - T)x = 0$$

  - $\mathbf{1}^t A = 0,\ \text{span}(\{x\}) = \text{null}_r(A),\ \text{spec}_r(A) \subset \mathcal{U}_1(1)$

Power Method

- Iterative method for computation of steady-state solution

$$x^{k+1} = Tx^k = (I - A)\,x^k = x^k - Ax^k$$

- Power method for $T \Leftrightarrow$ Richardson iteration for A
- With $1 = \lambda_1 > |\lambda_2| \geq \ldots \geq |\lambda_n| \geq 0$ eigenvalues of T

$$\|x^k - x\|_2 = c|\lambda_2|^k$$

$\Rightarrow$ Slow convergence for $\lambda_2 \approx 1$, yet fast reduction of components of x^k corresponding to $\lambda_i << 1$

Approximation of Eigenpair $Tx = \lambda_{max} x$ in a subspace $\mathcal{V}$

$$\text{Basis } v_1, \dots, v_k \text{ of } \mathcal{V}, \quad V = \begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix}$$

- Galerkin formulation

$$V^t T V y = \tilde{\lambda}_{max} V^t V y \quad \Rightarrow \tilde{x} = Vy \approx x,\ \tilde{\lambda}_{max} \approx \lambda_{max}$$

- Petrov-Galerkin formulation

$$W^t T V y = \tilde{\lambda}_{max} W^t V y \quad \Rightarrow \tilde{x} = Vy \approx x,\ \tilde{\lambda}_{max} \approx \lambda_{max}$$

Special case: Krylov subspaces $\mathcal{V} = \mathcal{K}_k$

- Orthonormal basis $v_1, \dots, v_k$, $V^t V = I$
- Bi-orthogonal basis $v_1, \dots, v_k$ and $w_1, \dots, w_k$, $W^t V = I$

Ingredients of Algebraic Multigrid (Needed, Given)

- Sparse linear system of equations $Au = f$
- Hierarchy of sparse systems of linear equations $A_l u_l = f_l$
- Appropriate smoothing iterations S_l
- Definition of restriction and interpolation

$$R_l^{l+1} : V_l \rightarrow V_{l+1} \quad \text{and} \quad P_{l+1}^l : V_{l+1} \rightarrow V_l$$

- Definition of operator hierarchy $A = A_0, A_1, \ldots, A_L$

$$A_{l+1} = R_l^{l+1} A_l P_{l+1}^l$$

$\Rightarrow$ Columns of P_{l+1}^l form sparse basis for subspace V_{l+1} in V_l complementary to the smoother

Least Squares Interpolation

Computation of interpolation weights $(p_i)_j, \quad i \in \mathcal{F}, j \in \mathcal{C}_i$

$$\mathcal{L}_{\mathcal{C}_i}(p_i) = \sum_{s=1}^{k} \omega_s \big(u_i^{(s)} - \sum_{j \in \mathcal{C}_i} (p_i)_j u_j^{(s)}\big)^2 \rightarrow \min_{p_i}$$

- Test vectors $u^{(1)}, \dots, u^{(k)} \in \mathbb{R}^n$,
- Set of interpolatory points $\mathcal{C}_i \subset \mathcal{C}$ for $i \in \mathcal{F}$
- Weights $\omega_s \in \mathbb{R}^+$ to bias LS fit, e.g., $\omega_s = \left(\frac{\|u^{(s)}\|}{\|Au^{(s)}\|}\right)^2$
- Splitting of variables $\Omega = \mathcal{F} \cup \mathcal{C}$, Interpolation P from $\mathcal{C}$ to Ω,

$$P = \begin{pmatrix} P_{fc} \\ I \end{pmatrix}, \quad p_{ij} \neq 0, i \in \mathcal{F}, j \in \mathcal{C}_i$$

Algebraic Multigrid Eigensolver — Bootstrap Setup

- Test vectors initially positive random, slightly smoothed

$$S^\eta u^{(1)}, \ldots, S^\eta u^{(k)}$$

- MG hierarchy with interpolation P^l_{l+1}, averaging restriction R^{l+1}_l $(\mathbf{1}^t R^{l+1}_l = \mathbf{1}^t)$

$$\begin{aligned} A_{l+1} &= R^{l+1}_l A_l P^l_{l+1} \\ B_{l+1} &= R^{l+1}_l B_l P^l_{l+1} \end{aligned}$$

Bootstrap Multigrid Eigensolver

if Coarsest level **then**

$\mathbf{V}_L = \{v_i^{(L)}\}_{i=1,\ldots,k_v}$

$A_L v_i^{(L)} = \lambda_i^{(L)} B_L v_i^{(L)}$

else

$v_i^{(l)} = P^l_{l+1} v_i^{(l+1)}$

$\lambda_i^{(l)} = \lambda_i^{(l+1)}$

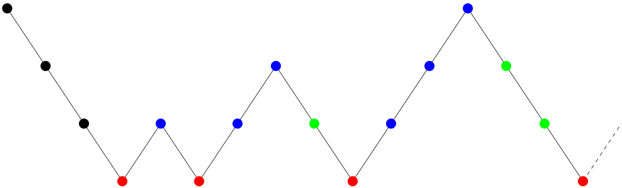
Relax $\left(A_l - \lambda_i^{(l)} B_l\right) v_i^{(l)} = 0$

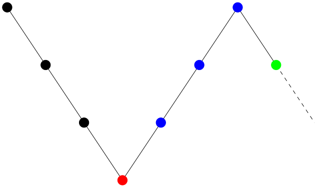
Update $\lambda_i^{(l)} = \frac{\langle A_l v_i^{(l)}, v_i^{(l)} \rangle_2}{\langle B_l v_i^{(l)}, v_i^{(l)} \rangle_2}$

end if

- New approximation of steady-state vector, enhancement of TV set, quality control of MG hierarchy

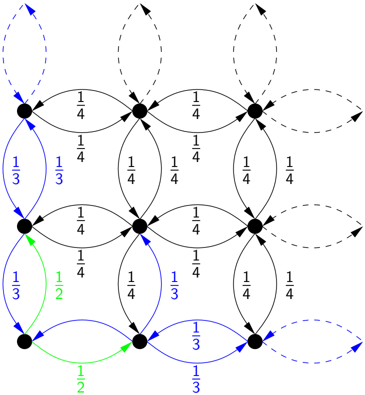
Bootstrap Multigrid Eigensolver Cycling Strategies





- ● Relax on $Au = 0$, $u \in \mathbf{U}$
- ● Compute $\mathbf{V}$, s.t., $Av = \lambda Bv$, $v \in \mathbf{V}$
- ● Relax on $Av = \lambda Bv$, $v \in \mathbf{V}$
- ● Relax on $Au = 0$, $u \in \mathbf{U}$ and $Av = \lambda Bv$, $v \in \mathbf{V}$

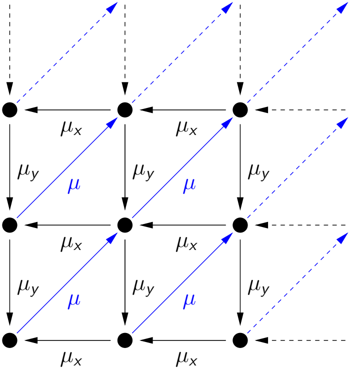
Uniform 2D Network



N	17	33	65	129
MLE	10	9	10	11
pArnoldi	7_V	8_V	10_V	9_{V^2}

- Accuracy 10^{-8}
- ω-Jacobi smoother with $\omega = .7$
- TV sets $|\mathbf{U}| = |\mathbf{V}| = 6$
- $V(2,2)$-cycle, caliber 2
- Coarsest grid 5×5

Tandem Queueing Network

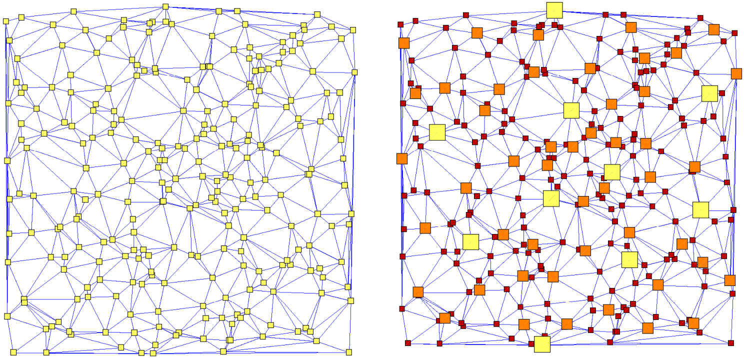


$$\mu = \frac{11}{31}, \quad \mu_x = \frac{10}{31}, \quad \mu_y = \frac{10}{31}$$

N	17	33	65	129
MLE	8	8	8	8
pArnoldi	6_V	6_V	6_V	7_V

- Accuracy 10^{-8}
- ω-Jacobi smoother with $\omega = .7$
- TV sets $|\mathbf{U}| = |\mathbf{V}| = 6$
- V(2, 2)-cycle, caliber 2
- Coarsest grid 5×5

Random Planar Graph



(a) Random planar graph, $n = 256$ (b) Coarsening, $n = 256$

$$p(j \to i) = d_{out}(j)^{-1}$$

Random Planar Graph

N	256	512	1024	2048
MLE	15	20	20	20
pArnoldi	8_V	10_V	10_V	11_V

- Accuracy 10^{-8}
- ω-Jacobi smoother with $\omega = .7$
- TV sets $|\mathbf{U}| = |\mathbf{V}| = 6$, caliber 2
- V$(2,2)$-cycle, 3-grid method

Conclusions and Outlook

Conclusions

- Algebraic Multigrid Eigensolver
- Reuse of AMG hierarchy as a "preconditioner"
- Promising and scaling results for simple Markov Chains

Outlook

- More complex Markov Chains
- Theory for LS interpolation and MG Eigensolver
- Efficient (parallel) implementation of Bootstrap AMG

Thank you for your attention!