Split Bezoutians and inverses of symmetric Toeplitz or centrosymmetric Toeplitz-plus-Hankel matrices

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$\mathbb{F}$ – field with a characteristic $\neq 2$,

$$J_n = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}, \quad \ell_n(t) = \left[\ t^j\ \right]_{j=0}^{n-1},\ t \in \mathbb{F}$$

Matrix language	$\leftrightarrow$	**Polynomial language**
$\mathbf{x} = \left[\ x_i\ \right]_{i=1}^{n} \in \mathbb{F}^n$	$\leftrightarrow$	$\mathbf{x}(t) = \ell_n(t)^T \mathbf{x} = \sum_{i=1}^{n} x_i t^{i-1}$
$A = \left[\ a_{ij}\ \right]_{i=1,j=1}^{m\ \ n} \in \mathbb{F}^{m\times n}$	$\leftrightarrow$	$A(t,s) = \ell_m(t)^T A \ell_n(s) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} t^{i-1} s^{j-1}$

Definitions:

$\mathbf{x} \in \mathbb{F}^n$ symmetric, if $\mathbf{x} = J_n \mathbf{x}$.

$\mathbf{x} \in \mathbb{F}^n$ skewsymmetric, if $\mathbf{x} = -J_n \mathbf{x}$.

$A \in \mathbb{F}^{n\times n}$ centrosymmetric, if $A = J_n A J_n$.

1 Inversion formulas - general case

Hankel matrix

$$H_n = \left[\ h_{i+j}\ \right]_{i,j=0}^{n-1} = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \cdots & h_n \\ \vdots & & \iddots & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{bmatrix}$$

Hankel-Bezoutian: $n \times n$ matrix B , for which $\mathbf{p}, \mathbf{q} \in \mathbb{F}^{n+1}$ exist, such that

$$B(t,s) = \frac{\mathbf{p}(t)\mathbf{q}(s) - \mathbf{q}(t)\mathbf{p}(s)}{t-s}\,.$$

Notation: $B = \mathrm{Bez}_H(\mathbf{p}, \mathbf{q})$

Theorem 1 *The inverse of a (nonsingular) Hankel matrix is a (nonsingular) H-Bezoutian, and vice versa.*

Toeplitz matrix

$$T_n = \left[\ a_{i-j}\ \right]_{i,j=1}^{n} = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-n+1} \\ a_1 & a_0 & \cdots & a_{-n+2} \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}$$

Toeplitz Bezoutian: $n \times n$ matrix B , $\mathbf{p}, \mathbf{q} \in \mathbb{F}^{n+1}$ exist, such that

$$B(t,s) = \frac{\mathbf{p}(t)\mathbf{q}(s^{-1})s^n - \mathbf{q}(t)\mathbf{p}(s^{-1})s^n}{1-ts}\,, \quad \left(J_{n+1}\mathbf{p}\right)(t) = \mathbf{p}(t^{-1})t^n\,.$$

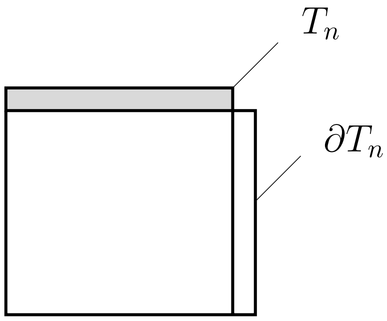
Notation: $B = \mathrm{Bez}_T(\mathbf{p}, \mathbf{q})$

Theorem 2 *The inverse of a (nonsingular) Toeplitz matrix is a (nonsingular) T-Bezoutian, and vice versa.*

$T_n J_n$ and $J_n T_n$ are Hankel matrices!

Restrict ourselves to the Toeplitz case

How to get $\mathbf{p}, \mathbf{q}$?



T_n nonsingular $\implies \dim \ker \partial T_n = 2$. ("$\Longleftarrow$" also true.)

Each basis $\{\mathbf{p}, \mathbf{q}\}$ of $\ker \partial T_n$ is called a fundamental system for T_n, since

$$c \,\mathrm{Bez}_T(\mathbf{p}, \mathbf{q}) = T_n^{-1} \qquad (c = \mathrm{const} \neq 0)\,.$$

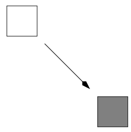
Normalization of $\mathbf{p}, \mathbf{q}$, so that $c = 1$!

How to get the entries of $T_n^{-1} = \mathrm{Bez}_T(\mathbf{p},\mathbf{q})$?

$$\mathrm{Bez}_T(\mathbf{p},\mathbf{q}) = \left[\ b_{ij}\ \right]_{i,j=1}^{n}, \quad \mathbf{p} = \left[\ p_i\ \right]_{i=0}^{n}, \quad \mathbf{q} = \left[\ q_i\ \right]_{i=0}^{n}$$

By recursion [Trench '64]

$$b_{ij} = b_{i-1,j-1} + p_{i-1}q_{n-j+1} - q_{i-1}p_{n-j+1}$$



By matrix representation [Gohberg, Semencul '72]

$$T_n^{-1} = \begin{bmatrix} p_0 & & & 0 \\ p_1 & p_0 & & \\ \vdots & & \ddots & \\ p_{n-1} & \cdots & \cdots & p_0 \end{bmatrix} \begin{bmatrix} q_n & q_{n-1} & \cdots & q_1 \\ & q_n & & \\ & & \ddots & \vdots \\ 0 & & & q_n \end{bmatrix} - \begin{bmatrix} q_0 & & & 0 \\ q_1 & q_0 & & \\ \vdots & & \ddots & \\ q_{n-1} & \cdots & \cdots & q_0 \end{bmatrix} \begin{bmatrix} p_n & p_{n-1} & \cdots & p_1 \\ & p_n & & \\ & & \ddots & \vdots \\ 0 & & & p_n \end{bmatrix}.$$

Toeplitz-plus-Hankel matrix

$$R_n = T_n + H_n$$

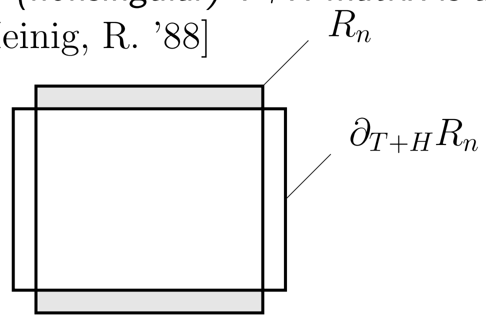
T+H-Bezoutian: $n \times n$ matrix B , for which $\mathbf{g}_i, \mathbf{f}_i \in \mathbb{F}^{n+2}$ $(i = 1, 2, 3, 4)$ exist, such that

$$B(t,s) = \frac{\sum_{i=1}^{4} \mathbf{g}_i(t)\mathbf{f}_i(s)}{(t-s)(1-ts)}\,.$$

Notation: $B = \mathrm{Bez}_{T+H}(\mathbf{g}_i, \mathbf{f}_i)_1^4$

Theorem 3 *The inverse of a (nonsingular) T+H matrix is a (nonsingular) $T+H$ Bezoutian, and vice versa.* [Heinig, R. '88]

How to get $\mathbf{g}_i, \mathbf{f}_i$?

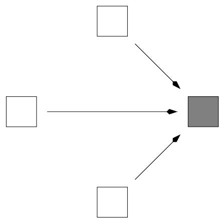


R_n nonsingular $\Longrightarrow$ $\dim\ker \partial_{T+H} R_n = 4$. ("$\Longleftarrow$" also true)

Normalization of any basis of $\ker \partial_{T+H} R_n \longrightarrow \{\mathbf{g}_i\}$
Normalization of any basis of $\ker \partial_{T+H} R_n^T \longrightarrow \{\mathbf{f}_i\}$

How to get the entries of $R_n^{-1} = \mathrm{Bez}_{T+H}(\mathbf{g}_i, \mathbf{f}_i)_1^4$?

By recursion: $\mathrm{Bez}_{T+H}(\mathbf{g}_i, \mathbf{f}_i)_1^4 = \left[\ b_{ij}\ \right]_{i,j=1}^{n}\ ,\ \left[\ a_{ik}\ \right]_{i,j=0}^{n+1} = \sum_{j=1}^{4} \mathbf{g}_j \mathbf{f}_j^T$



$$b_{ij} = b_{i-1,j-1} + b_{i+1,j-1} - b_{i,j-2} + a_{i,j-1}$$

By matrix representation

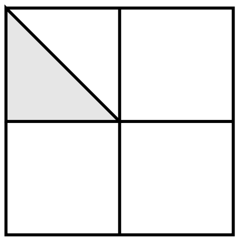
- complicated formula [Heinig, R. '89]
- but in case $\mathbb{F} = \mathbb{C}$ (or $\mathbb{F} = \mathbb{R}$) there are representations involving DFT's (or trigonometric transformations), diagonal and permutation matrices (e.g. [Heinig, R. '98], [Heinig, R. '00])

2 Inversion formulas - centrosymmetric case

We introduce special T+H - Bezoutians.
Let $\mathbf{p}, \mathbf{q} \in \mathbb{F}^{n+2}$ be both symmetric or both skewsymmetric. The $n \times n$ matrix B defined by

$$\boxed{B(t,s) = \frac{\mathbf{p}(t)\mathbf{q}(s) - \mathbf{q}(t)\mathbf{p}(s)}{(t-s)(1-ts)}}$$

is a polynomial in t and s and will be called split-Bezoutian of $\mathbf{p}$ and $\mathbf{q}$.
Notation: $\boxed{B = B_{\text{split}}(\mathbf{p}, \mathbf{q})}$
Useful symmetries: $B_{\text{split}}(\mathbf{p}, \mathbf{q})$ is a symmetric and centrosymmetric matrix.
If $\mathbf{p}$ and $\mathbf{q}$ are both symmetric (skewsymmetric), then all its columns and rows are symmetric (skewsymmetric). $\Longrightarrow$ split-Bezoutian of $(\pm)$-type.



Toeplitz matrix T_n centrosymmetric $\iff$ T_n symmetric

T+H matrix $R_n = T_n + H_n$ centrosymmetric $\implies$ R_n symmetric ("$\impliedby$" not true)

Theorem 4 *The inverse B of a symmetric Toeplitz matrix as well as of a centrosymmetric T+H matrix can be represented as sum*

$$B = B_+ + B_- \,,$$

where $B_\pm$ are split-Bezoutians of $(\pm)$-type. [Heinig, R. '03]

How to get the entries of $B_{\pm} = B_{\text{split}}(\mathbf{p}, \mathbf{q})$?

Demonstrate the result for the example of a split-Bezoutian of $(+)$-type of order 5

$$B_{+} = \left[\ b_{ij}\ \right]_{i,j=1}^{5} = B_{\text{split}}(\mathbf{p}, \mathbf{q})$$

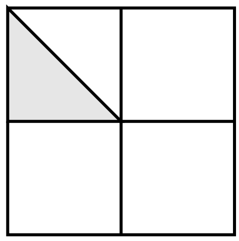
with

$$\begin{aligned}
\mathbf{p}(t) &= p_0(1+t^6) + p_1(t+t^5) + p_2(t^2+t^4) + p_3 t^3\,, \\
\mathbf{q}(t) &= q_0(1+t^6) + q_1(t+t^5) + q_2(t^2+t^4) + q_3 t^3\,.
\end{aligned}$$

Introduce

$$\boxed{a_{ij} = p_i q_j - q_i p_j}$$

- Recursion for the entries of the highlighted triangle B_{tri}



$$
\begin{aligned}
b_{11} &= a_{10} \\
b_{21} &= a_{20} \qquad & b_{22} &= a_{21} + b_{11} + b_{31} \\
& & &= a_{21} + a_{10} + a_{30} \\
b_{31} &= a_{30} \qquad & b_{32} &= a_{31} + 2b_{21} \qquad & b_{33} &= a_{32} + 2b_{22} - b_{31} \\
& & &= a_{31} + 2a_{20} & &= a_{32} + 2(a_{21} + a_{10}) + a_{30}
\end{aligned}
$$

- Matrix representation as linear combination of "elementary" split-Bezoutians of (+)-type

$$B_+^{i,j} = B_{\mathrm{split}}(t^i + t^{6-i}, t^j + t^{6-j}) \,,$$

namely

$$B_+ = a_{01}B_+^{0,1} + a_{02}B_+^{0,2} + a_{03}B_+^{0,3} + a_{12}B_+^{1,2} + a_{13}B_+^{1,3} + a_{23}B_+^{2,3} \,,$$

where

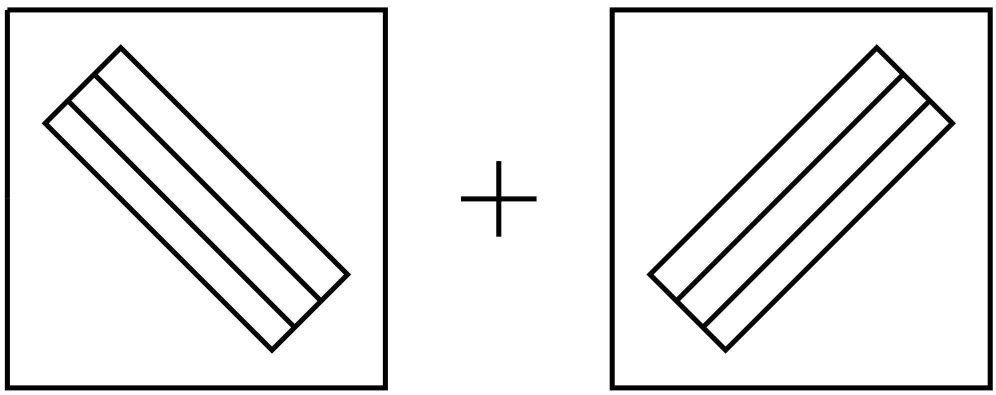
$$B_+^{0,1} = -(I_5 + J_5) = -\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} , \quad B_+^{0,3} = -\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} ,$$

$$B_+^{0,2} = -\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} ,$$

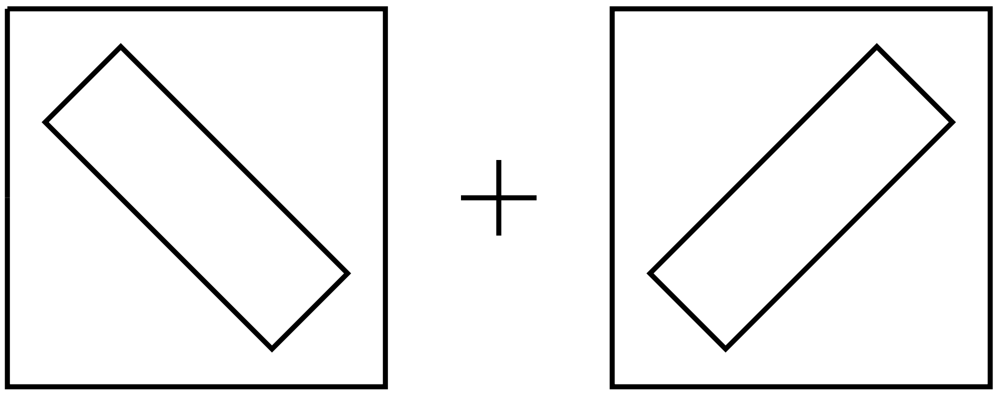
$$B_+^{1,2} = -\begin{bmatrix} 0 & & 0 \\ & I_3 & \\ 0 & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & 0 \\ & J_3 & \\ 0 & & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_+^{1,3} = -\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_+^{2,3} = -\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

General case: "elementary" split-Bezoutians



Modification: sparse matrices



- Matrix representation of Gohberg-Semencul type

The 3×3 leading principal submatrix of B_+ is of the form

$$\begin{bmatrix} q_0 & & \\ q_1 & q_0 & \\ q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & p_0 & p_1 \\ p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_2 \\ p_3 & p_2 & p_1 \end{bmatrix}$$

$$- \begin{bmatrix} p_0 & & \\ p_1 & p_0 & \\ p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & q_0 & q_1 \\ q_0 & q_1 & q_2 \\ q_1 & q_2 & q_3 \\ q_2 & q_3 & q_2 \\ q_3 & q_2 & q_1 \end{bmatrix}.$$

Let $H_5(\mathbf{p})$ and $Z_5(\mathbf{p})$ be the matrices

$$H_5(\mathbf{p}) = \begin{bmatrix} 0 & p_0 & p_1 & p_2 & p_3 \\ p_0 & p_1 & p_2 & p_3 & p_2 \\ p_1 & p_2 & p_3 & p_2 & p_1 \\ p_2 & p_3 & p_2 & p_1 & p_0 \\ p_3 & p_2 & p_1 & p_0 & 0 \end{bmatrix}, \quad Z_5(\mathbf{p}) = \begin{bmatrix} p_0 & 0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 & 0 \\ p_0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Denote by V_5 the matrix

$$V_5 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then

$$B_+ = Z_5(\mathbf{q})V_5H_5(\mathbf{p}) - Z_5(\mathbf{p})V_5H_5(\mathbf{q})$$

or

$$B_+ = Z_5(\mathbf{q})V_5T_5(\mathbf{p}) - Z_5(\mathbf{p})V_5T_5(\mathbf{q})\,,$$

where $T_5(\mathbf{p}) = J_5H_5(\mathbf{p})\,.$

- The general case

Theorem 5 *The $\ell \times \ell$ leading principal submatrix $B^{\ell}_{\pm}$ of the split Bezoutian of $(\pm)$-type $B_{\pm} = B_{\mathrm{split}}(\mathbf{p}, \mathbf{q})$ admits the representation*

$$B^{\ell}_{\pm} = T_{\ell}(\mathbf{q})\Lambda_{\ell,n}H_{n,\ell}(\mathbf{p}) - T_{\ell}(\mathbf{p})\Lambda_{\ell,n}H_{n,\ell}(\mathbf{q}) \,,$$

where $\ell = \left\lfloor \dfrac{n+1}{2} \right\rfloor$.

- Matrix-vector-multiplication

n even, $n = 2\ell$:

$$B_{\pm} = \begin{bmatrix} B^{\ell}_{\pm} & \pm B^{\ell}_{\pm} J_{\ell} \\ \pm J_{\ell} B^{\ell}_{\pm} & J_{\ell} B^{\ell}_{\pm} J_{\ell} \end{bmatrix}$$

$\mathbf{b} \in \mathbb{F}^n$, $\mathbf{b} = \mathbf{b}_+ + \mathbf{b}_-$, $\mathbf{b}_{\pm} = \frac{1}{2}(\mathbf{b} \pm J_n \mathbf{b})$

$$B_{\pm}\mathbf{b} = B_{\pm}(\mathbf{b}_+ + \mathbf{b}_-) = B_{\pm}\mathbf{b}_{\pm} = 2 \begin{bmatrix} B^{\ell}_{\pm}\mathbf{b}^{(\ell)}_{\pm} \\ \pm J_{\ell} B^{\ell}_{\pm}\mathbf{b}^{(\ell)}_{\pm} \end{bmatrix},$$

where $\mathbf{b}^{(\ell)}_{\pm}$ denotes the vector of the first ℓ entries of $\mathbf{b}_{\pm}$

(R. '09, Preprint 2009-15, Techn. Univ. Chemnitz, Fakultät für Mathematik])