# On Positive Definiteness of Singular Integral Operators

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joint work with
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$$\frac{q(x)}{\pi}\int_0^1 \frac{u(\xi)\,d\xi}{\xi - x}, \quad 0 < x < 1$$

# 1 Introduction

Does a Cauchy principal value integral commute with a usual integral?

The answer is:

Yes, under certain conditions!

Does a Cauchy principal value integral commute with a usual integral?

To formulate an example of suitable conditions we introduce the following notations:

- $\rho(x) = (1-x)^\gamma x^\delta =: \omega^{\gamma,\delta}(x)\,,\ \sigma(x) = (1-x)^{-\gamma} x^{-\delta} = \omega^{-\gamma,-\delta}(x)$

We assume that $\boxed{-1 < \gamma, \delta < 1}$.

- $\mathbf{L}^2_\rho = \left\{ u : (0,1) \to \mathbb{R} : \|u\|^2_\rho = \langle u, u \rangle_\rho < \infty \right\}\,,\ \langle u, v \rangle_\rho = \int_0^1 u(x)v(x)\rho(x)\,dx$

Remark that $\mathbf{X}^* = \mathbf{L}^2_\sigma$ is the dual space to $\mathbf{X} = \mathbf{L}^2_\rho$ with respect to the duality product $\langle u, v \rangle = \int_0^1 u(x)v(x)\,dx$ and that $\mathcal{J} : \mathbf{X} \longrightarrow \mathbf{X}^*$ with $(\mathcal{J}u)(x) = \rho(x)u(x)$ is the respective duality map.

- $(\mathcal{S}u)(x) := \frac{1}{\pi} \int_0^1 \frac{u(\xi)\,d\xi}{\xi - x} = \frac{1}{\pi} \lim_{\varepsilon \to +0} \left( \int_0^{x-\varepsilon} + \int_{x+\varepsilon}^1 \right) \frac{u(\xi)\,d\xi}{\xi - x}\,,\quad 0 < x < 1$

Does a Cauchy principal value integral commute with a usual integral?

**Lemma 1.1** *If* $u \in \mathbf{L}^2_\rho$ *and if* $v \in \mathbf{L}^2_\sigma$ *then*

$$\int_0^1 \int_0^1 \frac{u(\xi)\, d\xi}{\xi - x}\, v(x)\, dx = \int_0^1 u(\xi) \int_0^1 \frac{v(x)\, dx}{\xi - x}\, d\xi\,. \tag{1.1}$$

The proof of this Lemma is essentially based on the boundedness of the Cauchy singular integral operator $\mathcal{S} : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\rho$. Formula $(1.1)$ can be written in the form

$$\langle \mathcal{S}u, v\rangle = -\langle u, \mathcal{S}v\rangle$$

[Gachov, Muskhelischwili]

$$\boxed{\langle \mathcal{S}u, v\rangle = -\langle u, \mathcal{S}v\rangle \text{ for } u \in \mathbf{L}^2_\rho\,,\ v \in \mathbf{L}^2_\sigma}$$

**Corollary 1.2** *If* $u \in \mathbf{L}^2_\rho \cap \mathbf{L}^2_\sigma$ *then* $\langle \mathcal{S}u, u\rangle = 0$, *i.e., the operator*

$$\boxed{\mathcal{S} : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\sigma \textit{ with domain } D(\mathcal{S}) = \mathbf{L}^2_\rho \cap \mathbf{L}^2_\sigma}$$

*is monotone.*

[Askhabov, *Singular Integral equations and Equations of Convolution Type with Monotone Nonlinearity*, 2004]

## 2 Positive definite Cauchy singular integral operators

**Example 2.1** *For the operator* $\mathcal{S}_\alpha : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\sigma$ *defined by*

$$(\mathcal{S}_\alpha u)(x) = \frac{x^\alpha}{\pi} \int_0^1 \frac{u(\xi)\, d\xi}{\xi - x}, \quad 0 < x < 1\,,\ -1 < \alpha < 1\,,\ \alpha \neq 0\,,$$

*with domain* $D(\mathcal{S}_\alpha) = \left\{ u \in \mathbf{L}^2_\rho : \omega^\alpha u, \mathcal{S}_\alpha u \in \mathbf{L}^2_\sigma \right\}$ *, where* $\omega^\alpha(x) = x^\alpha$ *, we have*

$$\begin{aligned} \langle \mathcal{S}_\alpha u, u \rangle &= \langle \mathcal{S} u, \omega^\alpha u \rangle = \frac{1}{\pi} \int_0^1 \int_0^1 \frac{u(\xi)\, d\xi}{\xi - x}\, x^\alpha u(x)\, dx \\ &= \frac{1}{\pi} \int_0^1 \int_0^1 \frac{x^\alpha u(x)\, dx}{\xi - x}\, u(\xi)\, d\xi = \frac{1}{\pi} \int_0^1 \int_0^1 \frac{\xi^\alpha u(\xi)\, d\xi}{x - \xi}\, u(x)\, dx\,, \end{aligned}$$

*so that*

$$\langle \mathcal{S}_\alpha u, u \rangle = -\frac{1}{2\pi} \int_0^1 \int_0^1 \frac{x^\alpha - \xi^\alpha}{x - \xi}\, u(\xi) u(x)\, d\xi\, dx\,.$$

*Now, it yields*

$$
\begin{aligned}
\frac{x^\alpha - \xi^\alpha}{x-\xi} &= \frac{\alpha}{x-\xi}\int_\xi^x t^{\alpha-1}\,dt = \alpha\,\xi^{\alpha-1}\int_0^1 \left(1 + s\,\frac{x-\xi}{\xi}\right)^{\alpha-1} ds \\
&= \alpha\,\xi^{\alpha-1} F\left(1, 1-\alpha; 2; 1-\frac{x}{\xi}\right) = \frac{\sin(\pi\alpha)}{\pi}\,\xi^\alpha \int_0^\infty \frac{ds}{s^\alpha(1+s)(\xi+sx)} \\
&= \frac{\sin(\pi\alpha)}{\pi}\,\xi^\alpha x^\alpha \int_0^\infty \frac{dt}{t^\alpha(x+t)(\xi+t)}
\end{aligned}
$$

*with the Gauss hypergeometric function* $F = {}_2F_1$ .

$$\langle \mathcal{S}_\alpha u, u\rangle = -\frac{1}{2\pi}\int_0^1\int_0^1 \frac{x^\alpha - \xi^\alpha}{x-\xi}\, u(\xi)u(x)\, d\xi\, dx$$

$$\frac{x^\alpha - \xi^\alpha}{x-\xi} = \frac{\sin(\pi\alpha)}{\pi}\,\xi^\alpha x^\alpha \int_0^\infty \frac{dt}{t^\alpha (x+t)(\xi+t)}$$

*We see that the iterated integral*

$$\langle \mathcal{S}_\alpha |u|, |u|\rangle = -\frac{\sin(\pi\alpha)}{2\pi^2}\int_0^1\int_0^1\int_0^\infty \frac{\xi^\alpha x^\alpha |u(\xi)|\,|u(x)|}{t^\alpha (x+t)(\xi+t)}\, dt\, d\xi\, dx$$

*is finite. Hence, by the Theorem of Fubini-Tonelli it follows*

$$\langle \mathcal{S}_\alpha u, u\rangle = -\frac{\sin(\pi\alpha)}{2\pi^2}\int_0^\infty t^{-\alpha}\left(\int_0^1 \frac{x^\alpha u(x)\, dx}{x+t}\right)^2 dt\,,$$

*with* $\langle \mathcal{S}_\alpha u, u\rangle \le 0$ *for* $0 < \alpha < 1$ *and* $\langle \mathcal{S}_\alpha u, u\rangle \ge 0$ *for* $-1 < \alpha < 0$.

$$\langle \mathcal{S}_\alpha u, u\rangle = -\frac{\sin(\pi\alpha)}{2\pi^2}\int_0^\infty t^{-\alpha}\left(\int_0^1 \frac{x^\alpha u(x)\,dx}{x+t}\right)^2 dt$$

*If* $\langle \mathcal{S}_\alpha u, u\rangle = 0$ *then*

$$\int_0^1 \frac{x^\alpha u(x)\,dx}{x+t} = 0\,, \quad t>0\,,$$

*which implies, for* $t > 1$ ,

$$0 = \frac{1}{t}\int_0^1 x^\alpha u(x)\sum_{n=0}^\infty (-1)^n \frac{x^n}{t^n}\,dx = \sum_{n=0}^\infty (-1)^n \int_0^1 x^n x^\alpha u(x)\,dx\,t^{-(n+1)}\,.$$

*Consequently,* $\int_0^1 x^n x^\alpha u(x)\,dx = 0$ *for all* $n \in \mathbb{N}_0$ , *so that* $u(x) = 0$ *a.e. in* $(0,1)$ .

$$
\boxed{\begin{aligned} (\mathcal{S}_\alpha u)(x) &= \frac{x^\alpha}{\pi} \int_0^1 \frac{u(\xi)\, d\xi}{\xi - x}\,, \\ D(\mathcal{S}_\alpha) &= \left\{u \in \mathbf{L}^2_\rho : \omega^\alpha u, \mathcal{S}_\alpha u \in \mathbf{L}^2_\sigma\right\} \end{aligned}}
$$

**Proposition 2.2** *The operator $\mathcal{S}_\alpha : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\sigma$ is positive definite for $-1 < \alpha < 0$ and negative definite for $0 < \alpha < 1$.*

**Remark 2.3** *The operator $\mathcal{S}_1 : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\sigma$ is only negative semi definite, since*

$$
\langle \mathcal{S}_1 u, u \rangle = -\frac{1}{2\pi} \left( \int_0^1 u(x)\, dx \right)^2 .
$$

$$(\mathcal{S}_{\mu,\nu}u)(x) := \frac{(1-x)^{\mu}x^{\nu}}{\pi}\int_0^1 \frac{u(\xi)\,d\xi}{\xi - x}\,,\ -1<\mu,\nu<1$$

$$D(\mathcal{S}_{\mu,\nu}) = \left\{u \in \mathbf{L}^2_\rho : \omega^{\mu,\nu}u, \mathcal{S}_{\mu,\nu}u \in \mathbf{L}^2_\sigma\right\}$$

**Proposition 2.4** $\mathcal{S}_{\mu,\nu} : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\sigma$ *is positive definite for* $-1 < \nu \leq 0 \leq \mu < 1$ *and negative definite for* $-1 < \mu \leq 0 \leq \nu < 1$ *provided that* $|\mu| + |\nu| > 0$ *.*

$$
\left(\mathcal{L}_{c,d}u\right)(x) := \frac{q_0(x)}{\pi}\int_0^1 \frac{u(\xi)\,d\xi}{\xi - x}\,, \quad q_0(x) = \ln\frac{c}{x+d}\,,\ c>0,\ d\geq 0
$$

$$
D(\mathcal{L}_{c,d}) = \left\{u \in \mathbf{L}^2_\rho : q_0 u, \mathcal{L}_{c,d}u \in \mathbf{L}^2_\sigma\right\}
$$

**Proposition 2.5** *The operator $\mathcal{L}_{c,d} : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\sigma$ is positive definite.*

# 3 A general approach

Let $u \in \mathbf{L}^2_\rho$ and define

$$W(z) = u(z) + \mathbf{i}v(z) = \frac{1}{\pi \mathbf{i}} \int_0^1 \frac{u(\xi)\, d\xi}{\xi - z}\,, \quad \Im(z) > 0\,,$$

where $u, v : \{\Im(z) > 0\} \longrightarrow \mathbb{R}$. Moreover,

- $q(z) : \{\Im(z) > 0\} \longrightarrow \mathbb{C}$ holomorphic,
- $q : \{\Im(z) \geq 0\} \setminus \{x_1, \ldots, x_m\} \longrightarrow \mathbb{C}$ continuous,
- $qu,\ qv \in \mathbf{L}^2_\sigma\,,$
- $\ldots$

Finally, three cases of further assumptions on $q(x)$, $x \in \mathbb{R}$ are considered:

(I) $q(x) \in \mathbb{R}$ for $x > 0$,
$q(x) = q_1(x) + \mathbf{i}\, q_2(x)$ with $q_1(x) \in \mathbb{R}$, $q_2(x) \geq 0$ for $x < 0$,

(II) $q(x) > 0$ for $x > 1$ and

$$q(x) = \begin{cases} e^{\mathbf{i}\varphi}\, |q(x)| & : \quad 0 < x < 1\,, \\ e^{\mathbf{i}\beta}\, |q(x)| & : \ -\infty < x < 0\,, \end{cases} \tag{3.1}$$

where

$$\sin\varphi \geq 0\,, \quad \sin(\varphi - \beta) \geq 0\,,$$

(III) $q(x) < 0$ for $x > 1$ and $(3.1)$, where

$$\sin\varphi \geq 0\,, \quad \sin(\varphi - \beta) \leq 0\,.$$

Then it holds

$$\int_0^1 q(x)u(x)v(x)\,dx \geq 0$$

in case (I),

$$\int_0^1 |q(x)|u(x)v(x)\,dx \leq 0$$

in case (II),

$$\int_0^1 |q(x)|u(x)v(x)\,dx \geq 0$$

in case (III).

**Remark 3.1** *Due to the Sochozki-Plemelj formulas,*

$$\boxed{v(x) = -\frac{1}{\pi}\int_0^1 \frac{u(\xi)\,d\xi}{\xi - x}, \;\; 0 < x < 1}$$

**Example 3.2** *Let*

$$q(x) = \prod_{k=1}^{n} (x - x_k)^{\alpha_k}$$

*with* $n \in \mathbb{N}$, $-\infty < x_n < \ldots < x_1 \leq 0$, *and* $\alpha_k \in \mathbb{R}$. *The operator*

$$\boxed{\mathcal{Q} = q\mathcal{S} : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\sigma \text{ with domain } D(\mathcal{Q}) = \left\{ u \in \mathbf{L}^2_\rho : qu,\ \mathcal{Q}u \in \mathbf{L}^2_\sigma \right\}}$$

*is negative definite if*

$$0 < \alpha_k\,,\ k = 1, \ldots, n\,, \quad \text{and} \quad \sum_{k=1}^{n} \alpha_k < 1\,,$$

*and positive definite if*

$$\alpha_k < 0\,,\ k = 1, \ldots, n\,, \quad \text{and} \quad -1 < \sum_{k=1}^{n} \alpha_k\,.$$

# 4 Maximal monotone operators

$$\left(\mathcal{S}_{\mu,\nu}u\right)(x) := \frac{(1-x)^{\mu}x^{\nu}}{\pi}\int_0^1 \frac{u(\xi)\,d\xi}{\xi - x}\,,\ -1<\mu,\nu<1$$

$$D(\mathcal{S}_{\mu,\nu}) = \left\{u \in \mathbf{L}^2_\rho : \omega^{\mu,\nu}u, \mathcal{S}_{\mu,\nu}u \in \mathbf{L}^2_\sigma\right\}$$

**Recall**: $\mathcal{S}_{\mu,\nu} : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\sigma$ *is positive definite for* $-1<\nu\le 0\le \mu<1$ *and negative definite for* $-1<\mu\le 0\le\nu<1$ *provided that* $|\mu|+|\nu|>0$.

Let us consider a situation, where $D(\mathcal{S}_{\mu,\nu}) \neq \mathbf{L}^2_\rho$ : $\boxed{\gamma \le \mu \text{ and } \nu < \delta}$

$$D(\mathcal{S}_{\mu,\nu}) = \left\{\begin{array}{cc} \mathbf{L}^2_{\omega^{\gamma,2\nu-\delta}} & : \ -1 < 2\nu-\delta \\ \left\{u \in \mathbf{L}^2_{\omega^{\gamma,2\nu-\delta}} : \displaystyle\int_0^1 \frac{u(x)}{x}\,dx = 0\right\} & : \ 2\nu-\delta < -1 \end{array}\right\}$$

**Lemma 4.1** *A monotone operator* $\mathcal{T} : \mathbf{L}^2_\rho \longrightarrow \mathbf{L}^2_\sigma$ *is maximal monotone if and only if the equation*

$$\lambda\rho(x)u(x) + (\mathcal{T}u)(x) = f(x)$$

*has a solution* $u \in D(\mathcal{T})$ *for all* $\lambda > 0$ *and all* $f \in \mathbf{L}^2_\sigma$.

[Pascali, Sburlan, '78]

In our situation:

$$\lambda(1-x)^\gamma x^\delta u(x) + \frac{(1-x)^\mu x^\nu}{\pi} \int_0^1 \frac{u(\xi)\, d\xi}{\xi - x} = f(x)\,, \quad 0 < x < 1$$

$$\lambda(1-x)^{\gamma}x^{\delta}u(x)+\frac{(1-x)^{\mu}x^{\nu}}{\pi}\int_0^1\frac{u(\xi)\,d\xi}{\xi-x}=f(x)\,,\ \gamma\leq\mu \text{ and } \nu<\delta$$

$-1<2\nu-\delta$:

$$\lambda x^{\delta-\nu}u(x)+\frac{(1-x)^{\mu-\gamma}}{\pi}\int_0^1\frac{u(\xi)\,d\xi}{\xi-x}=(1-x)^{-\gamma}x^{-\nu}f(x)\in\mathbf{L}^2_{\omega^{\gamma,2\nu-\delta}}$$

- $\gamma\leq\mu$ and $\nu<\delta$ :

  The operator $\mathcal{S}_{\mu,\nu}:\mathbf{L}^2_{\rho}\longrightarrow\mathbf{L}^2_{\sigma}$ is maximal monotone if

  $$-2<2\nu-\delta<-1 \quad \text{or} \quad -1<2\nu-\delta$$

  and not maximal monotone if

  $$2\nu-\delta\leq-2$$