Compact Fourier Analysis for Multigrid Methods based on the Block Symbol

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Outline

1. Multigrid
2. Toeplitz Matrices and Generating Functions
3. Main results
4. Numerical results
5. Summary
Two-Grid Correction Scheme

Solution of the linear system $Ax = b$

1. Relax $\nu_1$ times on $A^h x^h = b^h$ on $\Omega^h \rightarrow$ approximation $v^h$

2. Coarse grid correction:
   - Compute the residual $r^h = b^h - A^h v^h$ and restrict it to the coarse grid $\rightarrow r^{2h}$
   - Solve $A^{2h} e^{2h} = r^{2h}$ on $\Omega^{2h} \rightarrow$ approximation $e^{2h}$
   - Interpolate the coarse-grid error to the fine grid
   - Correct the approximation $v^h \leftarrow v^h + e^{2h}$

3. Relax $\nu_2$ times on $A^h x^h = b^h$ on $\Omega^h \rightarrow$ approximation $v^h$
The matrices that have to be considered are

- **coarse grid correction**

  $$CGC = I - PA^{-1}_c P^T A$$

  $P$: prolongation,
  $P^T$: restriction,
  $A_c = P^T AP$ (Galerkin operator)

- **post- and presmoothing**

  $$S_L = I - M_L^{-1} A \quad \text{and} \quad S_R = I - M_R^{-1} A$$

  smoother $M$:  $$x_{k+1} = x_k + M^{-1}(b - Ax_k)$$
error reduction of a two-grid step

\[ TGS = S_L \cdot CGC \cdot S_R = (I - M_l^{-1}A)^{\nu_2} \cdot (I - PA_c^{-1}P^T A) \cdot (I - M_r^{-1}A)^{\nu_1} \]
Toeplitz Matrices and scalar generating functions

\[ T_n = T_n(f) = \begin{pmatrix}
  t_0 & t_{-1} & \cdots & \cdots & t_{1-n} \\
  t_1 & t_0 & t_{-1} & \cdots & \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  t_{n-1} & \cdots & \cdots & t_1 & t_0
\end{pmatrix} \]

Scalar generating function or symbol:

\[ f(x) = \sum_{j=-\infty}^{\infty} t_j e^{ijx} \]
Block Toeplitz Matrices and generating matrix functions

\[ T_n = T_n(f) = \begin{pmatrix}
T_0 & T_{-1} & \ldots & \ldots & T_{1-n} \\
T_1 & T_0 & T_{-1} & \vdots & \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & T_{-1} & \\
T_{n-1} & \ldots & \ldots & T_1 & T_0
\end{pmatrix} \]

Generating matrix function or block symbol:

\[ F(x) = \sum_{j=-\infty}^{\infty} T_j e^{ijx} \]
1D Model Problem

\[-u''(x) = f(x) \quad \text{for } x \in (0, 1),\]
\[u(0) = u(1) = 0.\]

Discretization with stencil \([-1 \ 2 \ -1]\) → linear system \(Ax = b\),
\[A = \text{tridiag}(-1, 2, -1)\]
→ Scalar symbol: \(- e^{ix} + 2 - e^{-ix} = 2(1 - \cos x)\)
Goal: write two-grid step in symbol → Fourier Analysis

Two classes of grid points in multigrid:
- grid points that appear also on the coarse level
- grid points that are only fine, but non-coarse

These two classes of grid points can be modeled by Block Symbols
Multigrid Toeplitz Matrices and Generating Functions

Main results
Numerical results
Summary

coarse noncoarse coarse noncoarse ...

\[
\begin{pmatrix}
2 & -1 \\
-1 & 2 \\
-1 & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 & -\alpha \\
-\bar{\alpha} & 2
\end{pmatrix}
\]
Block symbol:

\[
\begin{pmatrix}
2 & -1 \\
-1 & 2 \\
-1 & 2 \\
\vdots & \vdots & \ddots & \vdots
\end{pmatrix}
\]

\[
F(x) = \begin{pmatrix}
0 & -1 \\
0 & 0
\end{pmatrix} e^{ix} + \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
-1 & 0
\end{pmatrix} e^{-ix} =
\]

\[
= \begin{pmatrix}
2 & -1 - e^{ix} \\
-1 - e^{-ix} & 2
\end{pmatrix} = \begin{pmatrix}
2 & -\alpha \\
-\bar{\alpha} & 2
\end{pmatrix}
\]
P: odd-even permutation (red-black ordering)

$$PAP^T = \begin{pmatrix}
2 & 2 & -1 & -1 \\
2 & 2 & -1 & -1 \\
\vdots & \ddots & \ddots & \ddots \\
-1 & -1 & 2 & 2
\end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix}
2 & -1 - e^{-ix} & -1 - e^{ix} \\
-1 - e^{-ix} & 2
\end{pmatrix}$$
Also

- smoother
- projection
- coarse grid problem
- coarse grid correction
- two-grid step

can be described by means of generating functions

Example:

Gauss-Seidel smoother $\rightarrow \begin{pmatrix} 2 & -e^{ix} \\ -1 & 2 \end{pmatrix}$
2D Model problem

\[-u_{xx} - u_{yy} = f(x, y) \text{ for } x, y \in D = (0, 1) \times (0, 1),\]
\[u_{\partial D} = 0.\]

Discretization: 5-point stencil

\[
\begin{bmatrix}
  -1 & 4 & -1 \\
  -1 & 4 & -1 \\
  -1 & 4 & -1
\end{bmatrix}.
\]

Matrix: \( A = \text{tridiag}(-I, A_1, -I), \) where \( A_1 = \text{tridiag}(-1, 4, -1) \)

\[\iff A = A_2 \otimes I + I \otimes A_2, \text{ where } A_2 = \text{tridiag}(-1, 2, -1)\]
Multigrid Toeplitz Matrices and Generating Functions

Main results
Numerical results
Summary

\[
\begin{pmatrix}
4 & -1 & & & \\
-1 & \ddots & \ddots & & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & -1 \\
-1 & \ddots & \ddots & \ddots & 4 \\
\end{pmatrix}
\begin{pmatrix}
-1 & & & & \\
& \ddots & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & \ddots \\
\end{pmatrix}
\begin{pmatrix}
\cdot & & & & \\
& \ddots & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & \ddots \\
\end{pmatrix}
\]
Scalar symbol: $f(x, y) = 2(2 - \cos x - \cos y)$

Block symbol: $F(x, y) = \begin{pmatrix}
4 & -\alpha & -\beta & 0 \\
-\bar{\alpha} & 4 & 0 & -\beta \\
-\bar{\beta} & 0 & 4 & -\alpha \\
0 & -\bar{\beta} & -\bar{\alpha} & 4
\end{pmatrix}$,

where $\alpha = 1 + e^{ix}$, $\beta = 1 + e^{iy}$
A two-grid method is considered to be a **direct (exact) solver** if the total error is removed after one iteration.

- \( TGS = S_L \cdot CGC \cdot S_R = 0 \)
- Sufficient that \( S_L \cdot CGC = 0 \) or \( CGC \cdot S_R = 0 \)
- the smoother and the projection interact in such a manner that the range of the one matrix is in the nullspace of the other
- the actually iterative MG solver degenerates to a non-iterative, direct method
Theorem (Huckle, 2008)

If the symbol $TGS \equiv 0$, then also the two-grid error reduction for the original problem is zero up to a low rank term.
Main results (Huckle/K. 2009)

Theorem

Let $b_{1,P}$ be a given prolongation. Multigrid is a direct solver, when a presmoother $M_R$ is used, if and only if

$$M_R = F + (Fb_{1,P})d^H,$$

where $d$ is an arbitrary vector.
Theorem

Let $b_{1,R}$ be a given restriction. Multigrid is a direct solver, when a postsmoother $M_L$ is used, if and only if

$$M_L = F + cb_{1,R}^H F,$$

where $c$ is an arbitrary vector.
Definition

Subblock smoother \( SF(x, y) = \begin{pmatrix}
4 & 0 & 0 & 0 \\
-\bar{\alpha} & 4 & 0 & -\beta \\
-\bar{\beta} & 0 & 4 & -\alpha \\
0 & -\bar{\beta} & -\bar{\alpha} & 4
\end{pmatrix} \) \( SF = F + uv^H \), where

\[ u = \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}^T \quad \text{and} \quad v = \begin{pmatrix} 0 & -\alpha & -\beta & 0 \end{pmatrix}. \]
**Table:** Smoothing factors for various relaxations. The model problem is discretized with the 5-point stencil $A_5$.

<table>
<thead>
<tr>
<th>relaxation</th>
<th>smoothing factor</th>
<th>smoothing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$-JAC, $\omega = 1$</td>
<td>1</td>
<td>No</td>
</tr>
<tr>
<td>$\omega$-JAC, $\omega = 0.5$</td>
<td>0.75</td>
<td>Unsatisfactory</td>
</tr>
<tr>
<td>$\omega$-JAC, $\omega = 0.8$</td>
<td>0.6</td>
<td>Acceptable</td>
</tr>
<tr>
<td>GS-LEX</td>
<td>0.5</td>
<td>Good</td>
</tr>
<tr>
<td>GS-RB</td>
<td>0.25</td>
<td>Very good</td>
</tr>
<tr>
<td>subblock</td>
<td>0.0732</td>
<td>Excellent</td>
</tr>
</tbody>
</table>
Sparse approximations of the Galerkin coarse grid operator

- Multigrid algorithms with coarsening based on the Galerkin principle may lead to efficient solvers
- Disadvantage: produces coarse grid matrices that become thicker in every next grid

E.g.

5-point stencil
\[
\begin{bmatrix}
-1 & 4 & -1 \\
-1 & -1 & -1 \\
\end{bmatrix}
\]

9-point stencil
\[
\begin{bmatrix}
-1 & -1 & -1 \\
-1 & 8 & -1 \\
-1 & -1 & -1 \\
\end{bmatrix}
\]
Idea:
A and $A_c$ must be the same up to a constant factor
→ consider sparse approximations of $f_c$ of the form $\frac{f}{g}$,
g trigonometric polynomial

$$A_c = G^{-1} A, \quad A_c e = r \iff G^{-1} A e = r \iff A e = G r.$$ 

Benefit:
The coarse grid system is similar to the fine
→ practicable algorithm.
Purpose:
Find

\[
\min_{g} \left\| g - \frac{f}{f_c} \right\|_2^2
\]

Example.

\[
g_1(x, y) = a_0 + 2a_1(\cos x + \cos y) \\
= a_0 + a_1 e^{ix} + a_1 e^{-ix} + a_1 e^{iy} + a_1 e^{-iy}
\]
Multigrid Toeplitz Matrices and Generating Functions

<table>
<thead>
<tr>
<th>a₀</th>
<th>a₁</th>
<th></th>
<th>a₁</th>
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<tbody>
<tr>
<td>a₁</td>
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<td>⋱</td>
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</tbody>
</table>

Summary
\[
\min_{a_0, a_1} \left\| g_1 - \frac{f}{f_c} \right\|_2^2 = \min_{a_0, a_1} \int_0^\pi \int_0^\pi \left( a_0 + 2a_1 (\cos x + \cos y) - \frac{f}{f_c} \right)^2 \, dx \, dy
\]
Also:

\[ g_2(x, y) = g_1(x, y) + a_2(\cos(x - y) + \cos(x + y)), \]
\[ g_3(x, y) = g_2(x, y) + a_3(\cos 2x + \cos 2y), \]
\[ g_4(x, y) = g_3(x, y) + a_4(\cos(2x - y) + \cos(x - 2y) + \cos(2x + y) + \cos(x + 2y)) \]

and the constant case \( g_0(x, y) = a_0. \)

Degree of \( g_k : d := k + 1 \)
Fiorentino/Serra 1996

For deriving functioning multigrid, the projector must be chosen among those vanishing at the mirror points and being nonzero at the origin.

**mirror points**: $(x_0, \pi - y_0)$, $(\pi - x_0, y_0)$, $(\pi - x_0, \pi - y_0)$

$(x_0, y_0)$: singularity of $f$
Scalar symbol: \( f(x, y) = 2(2 - \cos x - \cos y) \), \((x_0, y_0) = (0, 0)\)
mirror points: \((0, \pi), (\pi, 0), (\pi, \pi)\)

Definition

The full projection

\[ b_{\text{full}}(x, y) := f(x, y + \pi)f(x + \pi, y)f(x + \pi, y + \pi) \]
prolongation: full
smother: subblock
\( \nu \): smoother applications
\( n=50 \) grid points
\( d \): degree of approximating trigonometric polynomial

Table: Optimal norms of \( TGC(x, y) \).

<table>
<thead>
<tr>
<th>( d )</th>
<th>restriction</th>
<th>( \nu = 1 )</th>
<th>( \nu = 2 )</th>
<th>( \nu = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>trivial</td>
<td>0.0447</td>
<td>0.0446</td>
<td>0.0444</td>
</tr>
<tr>
<td>2</td>
<td>standard</td>
<td>0.06</td>
<td>0.0598</td>
<td>0.0596</td>
</tr>
<tr>
<td>3</td>
<td>constant</td>
<td>0.0180</td>
<td>0.0179</td>
<td>0.0178</td>
</tr>
<tr>
<td>4</td>
<td>standard</td>
<td>0.0219</td>
<td>0.0218</td>
<td>0.0217</td>
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<tr>
<td>5</td>
<td>trivial</td>
<td>0.0098</td>
<td>0.0062</td>
<td>0.0045</td>
</tr>
</tbody>
</table>
prolongation: full
smother: subblock
\( \nu \): smoother applications
\( n=50 \) grid points
\( d \): degree of approximating trigonometric polynomial

**Table:** Optimal spectral radii of \( TGC(x, y) \).

<table>
<thead>
<tr>
<th>( d )</th>
<th>restriction</th>
<th>( \nu = 1 )</th>
<th>( \nu = 2 )</th>
<th>( \nu = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>trivial</td>
<td>0.0316</td>
<td>0.0315</td>
<td>0.0314</td>
</tr>
<tr>
<td>2</td>
<td>standard</td>
<td>0.0425</td>
<td>0.0423</td>
<td>0.0421</td>
</tr>
<tr>
<td>3</td>
<td>trivial</td>
<td>0.0127</td>
<td>0.0127</td>
<td>0.0126</td>
</tr>
<tr>
<td>4</td>
<td>standard</td>
<td>0.0155</td>
<td>0.0154</td>
<td>0.0154</td>
</tr>
<tr>
<td>5</td>
<td>trivial</td>
<td>0.0066</td>
<td>0.0042</td>
<td>0.0031</td>
</tr>
</tbody>
</table>
Table: Norms and spectral radii of $TGC(x, y)$ with standard Galerkin coarsening, $n=40$ points.

<table>
<thead>
<tr>
<th>smoother</th>
<th>projection</th>
<th>norm</th>
<th>spectral radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>GS-RB</td>
<td>standard</td>
<td>0.0703</td>
<td>0.0311</td>
</tr>
<tr>
<td>subblock</td>
<td>standard</td>
<td>0.0409</td>
<td>0.0258</td>
</tr>
<tr>
<td>subblock</td>
<td>full</td>
<td>1.6024e-14</td>
<td>1.1269e-14</td>
</tr>
</tbody>
</table>
prolongation: full
restriction: standard
presmother: subblock
\( \nu \): smoother applications
\( n = 50 \) grid points
\( d = 5 \): degree of approximating trigonometric polynomial

**Table:** Norms and spectral radii of \( TGC(x, y) \).

<table>
<thead>
<tr>
<th>postsmoother</th>
<th>( \nu )</th>
<th>norm</th>
<th>spectral radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>subblock</td>
<td>1</td>
<td>0.0097</td>
<td>0.0065</td>
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<tr>
<td>subblock</td>
<td>4</td>
<td>0.0035</td>
<td>0.0024</td>
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<td>GS-LEX</td>
<td>1</td>
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<td>GS-LEX</td>
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<tr>
<td>GS-RB</td>
<td>1</td>
<td>0.0112</td>
<td>0.0075</td>
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<tr>
<td>GS-RB</td>
<td>4</td>
<td>0.0044</td>
<td>0.0031</td>
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<tr>
<td>Jac</td>
<td>1</td>
<td>0.1950</td>
<td>0.1199</td>
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<tr>
<td>Jac</td>
<td>4</td>
<td>0.0063</td>
<td>0.0044</td>
</tr>
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</table>
Future work

- Aggregation-based multigrid method
- Especially smoothed aggregation
- Derive sparse approximate inverse smoothers by means of scalar/block symbols
- Three-grid analysis
- Application on more general PDEs
- Use of numerical optimization methods/genetic algorithms for identifying multigrid components with optimal interaction


The speaker's attendance at this conference was sponsored by the Alexander von Humboldt Foundation

http://www.humboldt-foundation.de
Thank you very much for your attention!

ckravv.googlepages.com