Compact Fourier Analysis for Multigrid Methods based on the Block Symbol

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Outline

Two-Grid Correction Scheme

Solution of the linear system $Ax = b$

1. Relax ν_1 times on $A^h x^h = b^h$ on $\Omega^h \longrightarrow$ approximation v^h
2. Coarse grid correction:
   - Compute the residual $r^h = b^h - A^h v^h$ and restrict it to the coarse grid $\longrightarrow\ r^{2h}$
   - Solve $A^{2h} e^{2h} = r^{2h}$ on $\Omega^{2h} \longrightarrow$ approximation e^{2h}
   - Interpolate the coarse-grid error to the fine grid
   - Correct the approximation $v^h \longleftarrow v^h + e^{2h}$
3. Relax ν_2 times on $A^h x^h = b^h$ on $\Omega^h \longrightarrow$ approximation v^h

The matrices that have to be considered are

- coarse grid correction

$$CGC = I - PA_c^{-1}P^TA$$

P: prolongation,
P^T: restriction,
$A_c = P^TAP$ (Galerkin operator)

- post- and presmoothing

$$S_L = I - M_L^{-1}A \quad \text{and} \quad S_R = I - M_R^{-1}A$$

smoother M: $\quad x_{k+1} = x_k + M^{-1}(b - Ax_k)$

- error reduction of a two-grid step

$$TGS = S_L \cdot CGC \cdot S_R = (I - M_l^{-1}A)^{\nu_2} \cdot (I - PA_c^{-1}P^TA) \cdot (I - M_r^{-1}A)^{\nu_1}$$

Toeplitz Matrices and scalar generating functions

$$T_n = T_n(f) = \begin{pmatrix} t_0 & t_{-1} & \cdots & \cdots & t_{1-n} \\ t_1 & t_0 & t_{-1} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & \cdots & t_1 & t_0 \end{pmatrix}$$

Scalar generating function or symbol:

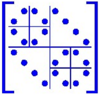
$$f(x) = \sum_{j=-\infty}^{\infty} t_j e^{ijx}$$

Block Toeplitz Matrices and generating matrix functions

$$T_n = T_n(f) = \begin{pmatrix} T_0 & T_{-1} & \cdots & \cdots & T_{1-n} \\ T_1 & T_0 & T_{-1} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & T_{-1} \\ T_{n-1} & \cdots & \cdots & T_1 & T_0 \end{pmatrix}$$

Generating matrix function or block symbol:

$$F(x) = \sum_{j=-\infty}^{\infty} T_j e^{ijx}$$



Model Problem

1D Model Problem

$$-u''(x) = f(x) \quad \text{for } x \in (0,1),$$
$$u(0) = u(1) = 0.$$

Discretization with stencil $[-1 \;\; 2 \;\; -1] \rightarrow$ linear system $Ax = b$,

$A = tridiag(-1, 2, -1)$

$\rightarrow$ Scalar symbol: $-e^{ix} + 2 - e^{-ix} = 2(1 - \cos x)$

Goal: write two-grid step in symbol $\rightarrow$ Fourier Analysis

Two classes of grid points in multigrid:

- grid points that appear also on the coarse level
- grid points that are only fine, but non-coarse

These two classes of grid points can be modeled by Block Symbols

$$
\begin{array}{c}
\begin{array}{cccc} \text{coarse} & \text{noncoarse} & \text{coarse} & \text{noncoarse} \ \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\
\left(\begin{array}{cc|cc|c}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
\hline
 & -1 & 2 & -1 & \\
 & & -1 & 2 & \\
\hline
 & & & & \ddots
\end{array}\right)
\end{array}
\begin{array}{l}
\leftarrow \text{coarse} \\
\leftarrow \text{noncoarse} \\
\leftarrow \text{coarse} \\
\leftarrow \text{noncoarse} \\
\quad \vdots
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
\begin{array}{cc} \text{coarse} & \text{noncoarse} \\ \downarrow & \downarrow \end{array} \\
\begin{pmatrix} 2 & -\alpha \\ -\bar{\alpha} & 2 \end{pmatrix}
\end{array}
$$

Block symbol:

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

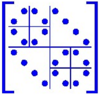
$$F(x) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} e^{ix} + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} e^{-ix} =$$

$$= \begin{pmatrix} 2 & -1 - e^{ix} \\ -1 - e^{-ix} & 2 \end{pmatrix} = \begin{pmatrix} 2 & -\alpha \\ -\bar{\alpha} & 2 \end{pmatrix}$$

P: odd-even permutation (red-black ordering)

$$PAP^T = \left(\begin{array}{cccc|cccc} 2 & & & & -1 & & & \\ & 2 & & & -1 & -1 & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & 2 & & & -1 & -1 \\ \hline -1 & -1 & & & 2 & & & \\ & -1 & \ddots & & & 2 & & \\ & & \ddots & -1 & & & \ddots & \\ & & & -1 & & & & 2 \end{array}\right)$$

$$\longleftrightarrow \begin{pmatrix} 2 & -1 - e^{ix} \\ -1 - e^{-ix} & 2 \end{pmatrix}$$



Also

- smoother
- projection
- coarse grid problem
- coarse grid correction
- two-grid step

can be described by means of generating functions

Example:

Gauss-Seidel smoother $\longrightarrow \begin{pmatrix} 2 & -e^{ix} \\ -1 & 2 \end{pmatrix}$

Short view on 2D

2D Model problem

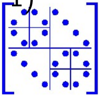
$$-u_{xx} - u_{yy} = f(x,y) \quad \text{for } x,\ y \in D = (0,1) \times (0,1),$$
$$u_{\partial D} = 0.$$

Discretization: 5-point stencil

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}.$$

Matrix: $A = tridiag(-I, A_1, -I)$, where $A_1 = tridiag(-1, 4, -1)$

$\longleftrightarrow A = A_2 \otimes I + I \otimes A_2$, where $A_2 = tridiag(-1, 2, -1)$



$$
\left(\begin{array}{cccc|cccc|cccc|c}
4 & -1 & & & -1 & & & & \cdot & & & & \\
-1 & \ddots & \ddots & & & \ddots & & & \cdot & & & & \\
 & \ddots & \ddots & -1 & & & \ddots & & \cdot & & & & \\
 & & -1 & 4 & & & & -1 & \cdot & & & & \\
\hline
-1 & & & & 4 & -1 & & & -1 & & & & \cdot \\
 & \ddots & & & -1 & \ddots & \ddots & & & \ddots & & & \cdot \\
 & & \ddots & & & \ddots & \ddots & -1 & & & \ddots & & \cdot \\
 & & & -1 & & & -1 & 4 & & & & -1 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

Scalar symbol: $f(x, y) = 2(2 - \cos x - \cos y)$

Block symbol: $$F(x, y) = \begin{pmatrix} 4 & -\alpha & -\beta & 0 \\ -\bar{\alpha} & 4 & 0 & -\beta \\ -\bar{\beta} & 0 & 4 & -\alpha \\ 0 & -\bar{\beta} & -\bar{\alpha} & 4 \end{pmatrix},$$

where $\alpha = 1 + e^{ix}, \quad \beta = 1 + e^{iy}$

Direct Solver

Definition

A two-grid method is considered to be a **direct (exact) solver** if the total error is removed after one iteration.

- $TGS = S_L \cdot CGC \cdot S_R = 0$
- Sufficient that $S_L \cdot CGC = 0$ or $CGC \cdot S_R = 0$
- the smoother and the projection interact in such a manner that the range of the one matrix is in the nullspace of the other
- the actually iterative MG solver degenerates to a non-iterative, direct method

Theorem (Huckle, 2008)

If the symbol $TGS \equiv 0$, then also the two-grid error reduction for the original problem is zero up to a low rank term.

Main results (Huckle/K. 2009)

Theorem

Let $b_{1,P}$ be a given prolongation. Multigrid is a direct solver, when a presmoother M_R is used, if and only if

$$M_R = F + (Fb_{1,P})d^H,$$

where d is an arbitrary vector.

Theorem

Let $b_{1,R}$ be a given restriction. Multigrid is a direct solver, when a postsmoother M_L is used, if and only if

$$M_L = F + c b_{1,R}^H F,$$

where c is an arbitrary vector.

Definition

Subblock smoother $SF(x,y) = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -\bar{\alpha} & 4 & 0 & -\beta \\ -\bar{\beta} & 0 & 4 & -\alpha \\ 0 & -\bar{\beta} & -\bar{\alpha} & 4 \end{pmatrix}$

$SF = F + uv^H$, where

$$u = \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}^T \text{ and } v = \begin{pmatrix} 0 & -\alpha & -\beta & 0 \end{pmatrix}.$$

Table: Smoothing factors for various relaxations. The model problem is discretized with the 5-point stencil A_5.

relaxation	smoothing factor	smoothing
ω-JAC, $\omega = 1$	1	No
ω-JAC, $\omega = 0.5$	0.75	Unsatisfactory
ω-JAC, $\omega = 0.8$	0.6	Acceptable
GS-LEX	0.5	Good
GS-RB	0.25	Very good
subblock	0.0732	Excellent

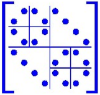
Sparse approximations of the Galerkin coarse grid operator

- Multigrid algorithms with coarsening based on the Galerkin principle may lead to efficient solvers
- Disadvantage: produces coarse grid matrices that become thicker in every next grid

e.g.

5-point stencil $\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} \rightarrow$

9-point stencil $\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$



Idea:
A and A_c must be the same up to a constant factor
$\longrightarrow$ consider sparse approximations of f_c of the form $\frac{f}{g}$,
g trigonometric polynomial

$$A_c = G^{-1}A, \qquad A_c e = r \Leftrightarrow G^{-1}Ae = r \Leftrightarrow Ae = Gr.$$

Benefit:
The coarse grid system is similar to the fine
$\longrightarrow$ practicable algorithm.

Purpose:

Find

$$\min_{g} \left\| g - \frac{f}{f_c} \right\|_2^2$$

Example.

$$\begin{aligned} g_1(x, y) &= a_0 + 2a_1(\cos x + \cos y) \\ &= a_0 + a_1 e^{ix} + a_1 e^{-ix} + a_1 e^{iy} + a_1 e^{-iy} \end{aligned}$$

$$
\left(\begin{array}{cccc|cccc|cccc|c}
a_0 & a_1 & & & a_1 & & & & \cdot & & & & \\
a_1 & \ddots & \ddots & & & \ddots & & & \cdot & & & & \\
 & \ddots & \ddots & a_1 & & & \ddots & & \cdot & & & & \\
 & & a_1 & 4 & & & & a_1 & \cdot & & & & \\
\hline
a_1 & & & & a_0 & a_1 & & & a_1 & & & & \cdot \\
 & \ddots & & & a_1 & \ddots & \ddots & & & \ddots & & & \cdot \\
 & & \ddots & & & \ddots & \ddots & a_1 & & & \ddots & & \cdot \\
 & & & a_1 & & & a_1 & a_0 & & & & a_1 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

$$\min_{a_0,a_1} \left\| g_1 - \frac{f}{f_c} \right\|_2^2 = \min_{a_0,a_1} \int_0^{\pi} \int_0^{\pi} \left(a_0 + 2a_1(\cos x + \cos y) - \frac{f}{f_c} \right)^2 dxdy$$

Also:

$$g_2(x,y) = g_1(x,y) + a_2(\cos(x-y) + \cos(x+y)),$$

$$g_3(x,y) = g_2(x,y) + a_3(\cos 2x + \cos 2y),$$

$$g_4(x,y) = g_3(x,y) + a_4(\cos(2x-y) + \cos(x-2y) + \cos(2x+y) + \cos(x+2y))$$

and the constant case $g_0(x,y) = a_0$.

Degree of g_k : $d := k+1$

Fiorentino/Serra 1996

For deriving functioning multigrid, the projector must be chosen among those vanishing at the mirror points and being nonzero at the origin.

mirror points: $(x_0, \pi - y_0)$, $(\pi - x_0, y_0)$, $(\pi - x_0, \pi - y_0)$

(x_0, y_0): singularity of f

Scalar symbol: $f(x,y) = 2(2 - \cos x - \cos y), \; (x_0, y_0) = (0,0)$

mirror points: $(0,\pi), \; (\pi,0), \; (\pi,\pi)$

Definition

The full projection

$$b_{full}(x,y) := f(x, y+\pi) f(x+\pi, y) f(x+\pi, y+\pi)$$

prolongation: full
smother: subblock
ν: smoother applications
n=50 grid points
d: degree of approximating trigonometric polynomial

Table: Optimal norms of $TGC(x, y)$.

d	restriction	$\nu = 1$	$\nu = 2$	$\nu = 3$
1	trivial	0.0447	0.0446	0.0444
2	standard	0.06	0.0598	0.0596
3	constant	0.0180	0.0179	0.0178
4	standard	0.0219	0.0218	0.0217
5	trivial	0.0098	0.0062	0.0045

prolongation: full

smother: subblock

ν: smoother applications

n=50 grid points

d: degree of approximating trigonometric polynomial

Table: Optimal spectral radii of $TGC(x, y)$.

d	restriction	$\nu = 1$	$\nu = 2$	$\nu = 3$
1	trivial	0.0316	0.0315	0.0314
2	standard	0.0425	0.0423	0.0421
3	trivial	0.0127	0.0127	0.0126
4	standard	0.0155	0.0154	0.0154
5	trivial	0.0066	0.0042	0.0031

Table: Norms and spectral radii of $TGC(x, y)$ with standard Galerkin coarsening, n=40 points.

smoother	projection	norm	spectral radius
GS-RB	standard	0.0703	0.0311
subblock	standard	0.0409	0.0258
subblock	full	1.6024e-14	1.1269e-14

prolongation: full
restriction: standard
presmother: subblock
ν: smoother applications
n=50 grid points
$d = 5$: degree of approximating trigonometric polynomial

Table: Norms and spectral radii of $TGC(x, y)$.

postsmoother	ν	norm	spectral radius
subblock	1	0.0097	0.0065
subblock	4	0.0035	0.0024
GS-LEX	1	0.0114	0.0082
GS-LEX	4	0.0045	0.0032
GS-RB	1	0.0112	0.0075
GS-RB	4	0.0044	0.0031
Jac	1	0.1950	0.1199
Jac	4	0.0063	0.0044

Future work

- Aggregation-based multigrid method
- Especially smoothed aggregation
- Derive sparse approximate inverse smoothers by means of scalar/block symbols
- Three-grid analysis
- Application on more general PDEs
- Use of numerical optimization methods/genetic algorithms for identifying multigrid components with optimal interaction

References

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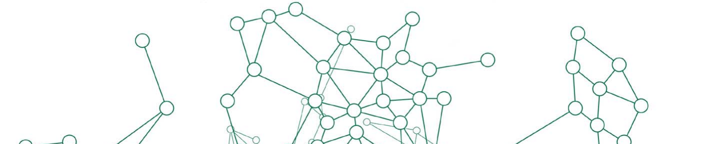

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