On a generalization of the concept of total positivity, based on the theory of cones

Olga Y. Kushel (Minsk, Belarus)

Theorem (Gantmacher–Krein). *If the matrix* $\mathbf{A}$ *of a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is totally positive (i.e. is positive together with all its minors of the* j*th order, where* $j = 2, \ldots, n$*), then all the eigenvalues of the operator* A *are positive, simple and different in modulus from each other:*

$$\rho(A) = \lambda_1 > \lambda_2 > \ldots > \lambda_n > 0.$$

Moreover, the first eigenvector, corresponding to the greatest in modulus eigenvalue λ_1 *is strictly positive, and the* j*th eigenvector* e_j*, corresponding to the* j*th in modulus eigenvalue* λ_j*, has exactly* $j-1$ *changes of sign and no zero coordinates.*

Theorem (Schoenberg). *If the matrix* $\mathbf{A}$ *of a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is totally positive, then the following inequality is true for each non-zero vector* $x \in \mathbb{R}^n$*:*

$$S^+(Ax) \leq S^-(x), \tag{1}$$

where $S^-(x)$ *is the number of sign changes in the sequence of the coordinates* $(x_1, \ldots, x_n)$ *of the vector* x*, with zero coordinates discarded.* $S^+(x)$ *is the maximum number of sign changes in the sequence* $(x_1, \ldots, x_n)$*, where zero coordinates are arbitrarily assigned values* ± 1*.*

Exterior square of the space $\mathbb{R}^n$

The space of all bilinear functionals on $(\mathbb{R}^n)' \times (\mathbb{R}^n)'$ is called a *tensor square* of the space $\mathbb{R}^n$ and denoted $\otimes^2\mathbb{R}^n$. Its elements are called *tensors*.

Let $x,\ y$ be arbitrary vectors from $\mathbb{R}^n$. Then the bilinear functional $x \otimes y : ((\mathbb{R}^n)' \times (\mathbb{R}^n)') \to \mathbb{R}$, which acts according to the rule

$$(x \otimes y)(f, g) = \langle x, f\rangle\langle y, g\rangle,$$

is called a *tensor product* of the vectors $x,\ y$. (Here the linear functionals $f, g \in (\mathbb{R}^n)'$ are considered as vectors from $\mathbb{R}^n$).

$$\otimes^2\mathbb{R}^n = \mathbb{R}^{n^2}$$

The subspace of all antisymmetric tensors (i.e. all the tensors φ, for which $\varphi(f,\ g) = -\varphi(g,\ f)$, where $f,\ g \in \mathbb{R}^n$) is called an *exterior square* of the space $\mathbb{R}^n$ and denoted $\wedge^2\mathbb{R}^n$.

Let $x,\ y$ be arbitrary vectors from $\mathbb{R}^n$. Then the bilinear functional $x \wedge y$, which is defined by the rule

$$x \wedge y = x \otimes y - y \otimes x,$$

is called an *exterior product* of the vectors $x,\ y$.

$$\wedge^2\mathbb{R}^n = \mathbb{R}^{C_n^2}$$

The set of all exterior products of the form $\{e_i \wedge e_j\}$, where $1 \leq i < j \leq n$, is a canonical basis in the space $\wedge^2\mathbb{R}^n$.

Let us define a map $\mathcal{A}$, which acts from the set of all 2-dimensional subspaces of $\mathbb{R}^n$ to the set of 1-dimensional subspaces (i.e. lines) of $\wedge^2\mathbb{R}^n$ according to the following rule:

$$\mathcal{A}(L) = \{t(x_1 \wedge x_2)\}_{t \in \mathbb{R}}, \tag{2}$$

where L is a 2-dimensional subspace from $\mathbb{R}^n$, $x_1,\ x_2$ are two arbitrary linearly independent vectors from L.

Conic sets in $\mathbb{R}^n$

A closed subset $K \subset \mathbb{R}^n$ is called a *proper cone*, if it is a convex cone (i.e. for any $x, y \in K,\ \alpha \geq 0$ we have $x + y,\ \alpha x \in K$), pointed ($K \cap (-K) = \{0\}$) and full ($\mathrm{int}(K) \neq \emptyset$).

A closed subset $T \subset \mathbb{R}^n$ is called *a cone of rank* k ($0 \leq k \leq n$), if for every $x \in T$, $\alpha \in \mathbb{R}$ the element $\alpha x \in T$ and there is at least one k-dimensional subspace and no higher dimensional subspaces in T.

Example. Let $M_k = \{x \in \mathbb{R}^n : S^-(x) \leq k - 1\}$, i.e. the set of all vectors in $\mathbb{R}^n$, which have no more than $k - 1$ sign changes in the sequence of their non-zero coordinates. Then M_k is a cone of rank k for every $k = 1, \ \ldots, \ n$.

Given two convex cones $K_1 \subset \mathbb{R}^n$ and $K_2 \subset \mathbb{R}^{C_n^2}$. Let us define the set $T(K_1, K_2) \subset \mathbb{R}^n$ by the following way:

$$T(K_1, K_2) = \\ = \overline{\{x \in \mathbb{R}^n : \exists\ k \in (K_1 \cup (-K_1)) \setminus \{0\},\ \text{for which}\ k \wedge x \in (K_2 \cup (-K_2)) \setminus \{0\}\}}. \qquad (3)$$

Lemma 1. *Let $n \geq 3$, $K_1 \subset \mathbb{R}^n$ and $K_2 \subset \wedge^2\mathbb{R}^n$ be two proper cones. Then the set $T(K_1, K_2)$, defined by formula (3), coincides with the closure of the set of all 2-dimensional subspaces $L \subset \mathbb{R}^n$, which satisfy the following conditions*

1. *The corresponding line $\mathcal{A}(L)$ belongs to $K_2 \cup (-K_2)$;*
2. *The intersection $L \cap K_1 \neq \{0\}$.*

Theorem 1. *Let K_1 be a proper cone in $\mathbb{R}^n$. Let K_2 be a proper cone in $\mathbb{R}^{C_n^2}$, which satisfy the following condition: $K_2 \subseteq K_2'$, where K_2' is a proper cone, which is spanned on the vectors of the form $e_i' \wedge e_j'$ $(1 \leq i, j \leq n)$, where $e_1', \ldots, e_n'$ are linearly independent vectors from $\mathbb{R}^n$. Then the set $T(K_1, K_2)$, if it is nonempty, is a cone of rank 2.*

Generalized strictly 2-totally positive operators

Let $K \subset \mathbb{R}^n$ be a proper cone. A linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is called *K-positive* or *positive with respect to the cone K* if $A(K \setminus \{0\}) \subseteq \text{int}(K)$. In the case of $K = \mathbb{R}^n_+$ K-positive operators are called simply *positive*.

Theorem (Generalized Perron). *Let a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ be K-positive with respect to a proper cone $K \subset \mathbb{R}^n$. Then:*

1. *The spectral radius $\rho(A) > 0$ is a simple positive eigenvalue of the operator A, different in modulus from the other eigenvalues.*

2. *The eigenvector x_1, corresponding to the eigenvalue $\lambda_1 = \rho(A)$, belongs to* $\text{int}(K)$.

4. *The functional x_1^*, corresponding to the eigenvalue $\lambda_1 = \rho(A)$, satisfy the inequality $x_1^*(x) > 0$ for every nonzero $x \in K$.*

A linear operator $\wedge^2 A$, which acts in the space $\wedge^2 \mathbb{R}^n$ according to the rule:

$$(\wedge^2 A)(x \wedge y) = Ax \wedge Ay,$$

is called the exterior square of the operator A.

If the operator A is given by a matrix $\mathbf{A} = \{a_{ij}\}_{i,j=1}^n$ in the basis $e_1, \ldots, e_n$, then the matrix of its exterior square $\wedge^2 A$ in the basis $\{e_i \wedge e_j\}$, where $1 \leq i < j \leq n$, coincides with the second compound matrix

$\mathbf{A}^{(2)}$ of the initial matrix $\mathbf{A}$, i.e. with the matrix which consists of all the minors of the second order $A\begin{pmatrix} i & j \\ k & l \end{pmatrix}$, where $1 \leq i < j \leq n,\ 1 \leq k < l \leq n$, of the initial $n \times n$ matrix $\mathbf{A}$.

A linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is called (K_1, K_2)-*strictly totally positive*, if the inclusion $A(K_1 \setminus \{0\}) \subseteq \mathrm{int}(K_1)$ is true for some proper cone $K_1 \subset \mathbb{R}^n$, and the inclusion $(A \wedge A)(K_2 \setminus \{0\}) \subseteq \mathrm{int}(K_2)$ is true for some proper cone $K_2 \subset \mathbb{R}^{C_n^2}$.

In the case, when $K_1 = \mathbb{R}^n_+$, $K_2 = \mathbb{R}^{C_n^2}_+$, the definition, given above, coincides with the classical definition of 2-strictly totally positive operator, given by F.R. Gantmacher and M.G. Krein.

Gantmacher–Krein theorem and variational diminishing property of generalized strictly 2-totally positive operators

Theorem 2. *Let a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be strictly* K_1, K_2*-totally positive. Then the following inclusion is true:*

$$A(T(K_1, K_2)) \subseteq T(K_1, K_2).$$

Theorem 3. *Let a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be strictly* K_1, K_2*-totally positive. Then the operator* A *has two positive simple eigenvalues, different in modulus from each other and from the rest of eigenvalues:*

$$0 \leq \ldots \leq |\lambda_3| < \lambda_2 < \lambda_1.$$

The first eigenvector x_1 *corresponding to the maximal eigenvalue* λ_1*, belongs to* $\mathrm{int}(K_1)$*. The second eigenvector* x_2*, corresponding to the second in modulus eigenvalue* λ_2*, belongs to* $\mathrm{int}(T(K_1, K_2) \setminus K_1)$*.*