On the Perron-Frobenius Theory for complex matrices

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Applied Linear Algebra
In honor of Hans Schneider
Novi Sad, Serbia, May 24-28, 2010

History

Theorem

The dominant eigenvalue of a matrix with positive entries is positive and the corresponding eigenvector could be chosen to be positive.

Theorem

The dominant eigenvalue of an irreducible nonnegative matrix is positive and the corresponding eigenvector could be chosen to be positive.

- O. Perron, *Zur Theorie der Matrizen*, Math. Ann. 64 (1907), 248–263.
- G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, S.-B. Preuss Acad. Wiss. Berlin (1912), 456–477.

Extension of the Perron-Frobenius theory to real matrices

The Perron-Frobenius theory may hold also in the case where the matrix has some absolutely small negative entries.

- How small could these entries be?
- What is their distribution?
- When such a matrix looses the Perron-Frobenius property?

Tarazaga et' al gave a partial answer to the first question by providing a sufficient condition for the symmetric matrix case.

Noutsos answered implicitly the above questions: Sufficient and necessary conditions were given and the Perron-Frobenius theory of nonnegative matrices was extended to the class of matrices that **possess the Perron-Frobenius property.**

- P. Tarazaga, M Raydan and A. Hurman, *Perron-Frobenius theorem for matrices with some negative entries*, Linear Algebra Appl. 328 (2001), 57–68.
- D. Noutsos. *On Perron-Frobenius property of matrices having some negative entries.* Linear Algebra Appl. 412 (2006), 132–153.

Extension of Perron-Frobenius theory to complex matrices

Could we extent the Perron-Frobenius theory to complex matrices???

If such an extension exists,
how the imaginary part of the matrix behaves?
Does it suffice to have absolutely small entries?

Our aim is:

- To define this extension.
- To characterize complex matrices that possess the Perron-Frobenius property by sufficient and necessary conditions.
- To introduce and study the class of Perron-Frobenius splitings.

Definition

A matrix $A \in \mathbb{C}^{n,n}$ possesses the Perron-Frobenius property if its dominant eigenvalue λ_1 is positive and the corresponding eigenvector $x^{(1)}$ is nonnegative.

Definition

A matrix $A \in \mathbb{C}^{n,n}$ possesses the strong Perron-Frobenius property if its dominant eigenvalue λ_1 is positive, simple $(\lambda_1 > |\lambda_i|,\ i = 2, 3, \cdots, n)$ and the corresponding eigenvector $x^{(1)}$ is positive.

Extension of the Perron-Frobenius theory – Type II

Definition

A matrix $A \in \mathbb{C}^{n,n}$ possesses the complex Perron-Frobenius property if its dominant eigenvalue λ_1 is positive and its corresponding eigenvector $x^{(1)}$ can be chosen so that $Rex^{(1)} \geq 0$. i.e., if $x^{(1)} = [x_1^{(1)}\ x_2^{(1)} \cdots\ x_n^{(1)}]^T$, then $Rex_j^{(1)} \geq 0$ for all $j = 1, 2, \ldots, n$.

Definition

A matrix $A \in \mathbb{C}^{n,n}$ possesses the strong complex Perron-Frobenius property if its dominant eigenvalue λ_1 is positive, simple, with $\lambda_1 > |\lambda_i|,\ i = 2, 3, \ldots, n$, and for the corresponding eigenvector $x^{(1)}$, there holds: $Rex^{(1)} > 0$, i.e., $Rex_j^{(1)} > 0$ for all $j = 1, 2, \ldots, n$.

Remark: Since the eigenvectors are invariant under scalar multiplication, the components $x_j^{(1)},\ j = 1, 2, \ldots, n$ can be assumed to be in any half-plane, via rotation.

Extension of the Perron-Frobenius theory – Type I

Lemma

Let $A \in \mathbb{C}^{n,n}$ have the Perron-Frobenius property with the eigenpair $(\rho(A), x)$. On writing $A^k = A_{k,1} + \mathrm{i}A_{k,2}$ for any positive integer k, where $A_{k,1}$ and $A_{k,2}$ are real matrices in $\mathbb{R}^{n,n}$, then $A_{k,2}x = 0$ for any k.

Proof: For the eigenpair $(\rho(A), x)$, we have

$$A^k x = A_{k,1}x + \mathrm{i}A_{k,2}x = (\rho(A))^k x \geq 0,$$

which implies that $A_{k,2}x = 0$.
For $k = 1$: $A_{1,2}x = ImAx = 0$.

Sufficient and necessary conditions – Type I

Theorem

For a matrix $A \in \mathbb{C}^{n,n}$, the following properties are equivalent:
i) Both matrices A and A^H possess the strong Perron-Frobenius property.
ii) There exists an integer $k_0 > 0$ such that $Re(A^k) > O$ for all $k \geq k_0$ and for the eigenvalues of A there holds
$|\lambda_1| > |\lambda_i|, \quad i = 2, 3, \ldots, n.$

Sufficient and necessary conditions – Type I

Example

Consider the matrix

$$A = \begin{pmatrix} -0.25 + 0.25i & 0.5 - 0.5i & 2.75 + 0.25i \\ 4.75 + 0.25i & 1.5 + 1.5i & -3.25 - 1.75i \\ 1.75 - 0.75i & 0.5 - 0.5i & 0.75 + 1.25i \end{pmatrix},$$

whose eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -2 + \mathrm{i}$, $\lambda_3 = 1 + 2\mathrm{i}$. Both matrices A and A^H possess the strong Perron-Frobenius property, with eigenvectors $x^{(1)} = (1\ 1\ 1)^T$ and ${y^{(1)}}^T = (0.5\ 0.25\ 0.25)$, respectively. Thus, it should exists $k_0 > 0$ such that $Re(A^k) > O$.

Example

By computing the normalized powers of A, i.e., $\frac{1}{\lambda_1^k}A^k$, we find that $\frac{1}{\lambda_1^4}A^4 =$

$$\begin{pmatrix} 0.4568 - 0.1481\mathrm{i} & 0.2716 + 0.0741\mathrm{i} & 0.2716 + 0.0741\mathrm{i} \\ 0.5432 + 0.1481\mathrm{i} & 0.1852 - 0.2222\mathrm{i} & 0.2716 + 0.0741\mathrm{i} \\ 0.5432 + 0.1481\mathrm{i} & 0.2716 + 0.0741\mathrm{i} & 0.1852 - 0.2222\mathrm{i} \end{pmatrix}.$$

We observe that $\frac{1}{\lambda_1^4}Re(A^4) > O$.

It is checked that $\frac{1}{\lambda_1^k}Re(A^k) > O$ for all $k \geq 4$, with

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k}A^k = x^{(1)}{y^{(1)}}^T = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0.25 & 0.25 \\ 0.5 & 0.25 & 0.25 \end{pmatrix}.$$

Sufficient and necessary conditions – Type II

Theorem

For a matrix $A \in \mathbb{C}^{n,n}$*, the following properties are equivalent:*
i) Both matrices A *and* A^H *possess the strong complex Perron-Frobenius property, with Perron-Frobenius eigenvectors* $x^{(1)}$ *and* $y^{(1)}$*, respectively, such that* ${y^{(1)}}^H x^{(1)} = 1$ *and* $Re(x^{(1)}{y^{(1)}}^H) > 0$.
ii) There exists an integer $k_0 > 0$ *such that* $Re(A^k) > O$ *for all* $k \geq k_0$ *and for the eigenvalues of* A *there holds* $|\lambda_1| > |\lambda_i|, \quad i = 2, 3, \ldots, n.$

Example

Consider the matrix $A =$

$$\begin{pmatrix} 0.7500 - 1.1250i & 0.5882 - 0.1471i & 1.0735 + 1.4191i \\ -0.5000 - 1.0000i & 2.1765 + 0.7059i & 2.1471 - 0.4118i \\ 2.7500 - 0.1250i & 0.5882 - 0.1471i & -0.9265 + 0.4191i \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -2 - \mathrm{i}$, $\lambda_3 = 1 + \mathrm{i}$,
Both matrices A and A^H possess the strong complex Perron-Frobenius property, with eigenvectors $x^{(1)} = (0.2353 - 0.0588i \; 0.4706 - 0.1176i \; 0.2353 - 0.0588i)^T$ and ${y^{(1)}}^H = (1 - 0.5i \; 1 + 0.5i \; 1 + 0.5i)$, respectively. Thus, it should exist an integer $k_0 > 0$ such that $Re(A^k) > O$ for all $k \geq k_0$.
It can be verified that $A^3 =$

$$\begin{pmatrix} 7.3456 - 8.1176i & 7.7941 + 1.1765i & 4.0662 + 5.7647i \\ 8.9412 - 13.7353i & 13.5882 + 4.3529i & 17.8824 + 5.0294i \\ 9.3456 + 2.8824i & 7.7941 + 1.1765i & 2.0662 - 5.2353i \end{pmatrix}$$

and that $Re(A^k) > O$, $k \geq 3$, with $\lim_{k\to\infty} \frac{1}{\lambda_1^k} A^k = x^{(1)} {y^{(1)}}^H$.

Weaker conditions – Type I

Theorem (Necessary conditions)

Assume that both the matrices $A \in \mathbb{C}^{n,n}$ and A^H possess the Perron-Frobenius property, with the dominant eigenvalue $\lambda_1 = \rho(A)$ being simple and $\lambda_1 > |\lambda_i|, \; i = 2, 3, \ldots, n$. Then,

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} A^k = x^{(1)} {y^{(1)}}^T \geq O.$$

Example

Consider the matrix

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} - \frac{1}{6}\mathrm{i} & \frac{1}{2} + \frac{1}{6}\mathrm{i} \\ \frac{3}{2} & \frac{1}{2} + \frac{1}{3}\mathrm{i} & -1 - \frac{1}{3}\mathrm{i} \\ \frac{3}{4} - \frac{1}{4}\mathrm{i} & \frac{1}{4} - \frac{1}{6}\mathrm{i} & \frac{5}{12}\mathrm{i} \end{pmatrix},$$

whose eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2} + \frac{1}{4}\mathrm{i}$, $\lambda_3 = \frac{1}{4} + \frac{1}{2}\mathrm{i}$.

Example

Both matrices A and A^H possess the Perron-Frobenius property, with eigenvectors $x^{(1)} = (1\ 1\ 1)^T$ and $y^{(1)^T} = (\frac{2}{3}\ \frac{1}{3}\ 0)$, respectively.

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} A^k = \lim_{k\to\infty} A^k = x^{(1)} y^{(1)^T} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

The vector sequence of the third column tends to the zero. It doesn't guarantee that A^k has necessarily a nonnegative real part. This is shown in the successive iterations of $a_k := 10^{13} \times \left(A^k\right)_3$, where $\left(A^k\right)_3$ denotes the third column of A^k.

$$a_{52} = \begin{pmatrix} 0 \\ 0 \\ 0.4 - 0.6i \end{pmatrix},\ a_{53} = \begin{pmatrix} 0.1 + 0.4i \\ -0.2 - 0.8i \\ -0.3 + 0.2i \end{pmatrix},\ a_{54} = \begin{pmatrix} -0.3 - 0.03i \\ 0.6 + 0.06i \\ -0.1 - 0.01i \end{pmatrix}$$

Weaker conditions – Type II

Theorem (Sufficient conditions)

Let $A \in \mathbb{C}^{n,n}$ be nonnilpotent and there exists $k_0 > 0$ such that $Re(A^k) \geq O$ for all $k \geq k_0$. Then, both matrices A and A^H possess the complex Perron-Frobenius property.

Theorem (Necessary conditions)

Let both the matrices $A \in \mathbb{C}^{n,n}$ and A^H possess the complex Perron-Frobenius property, with the dominant (largest in modulus) eigenvalue $\lambda_1 = \rho(A)$ being simple and $\lambda_1 > |\lambda_i|, \ i = 2, 3, \dots, n$. Then,

$$\lim_{k \to \infty} \frac{1}{\lambda_1^k} Re(A^k) \geq O.$$

Row sums and spectral radius

Theorem (Type I)

If $A^H \in \mathbb{C}^{n,n}$ possesses the Perron-Frobenius property, then either

$$\sum_{j=1}^{n} Re(a_{ij}) = \rho(A) \ \ \forall\ i = 1(1)n,$$

or

$$\min_i \left(\sum_{j=1}^{n} Re(a_{ij}) \right) \leq \rho(A) \leq \max_i \left(\sum_{j=1}^{n} Re(a_{ij}) \right).$$

Moreover, if A^H possesses the strong Perron-Frobenius property, then both inequalities are strict.

Analogous result is valid for column sums as well as for weighted row or column sums.

Monotonicity properties

Theorem (Type I)

If $A, B^H \in \mathbb{C}^{n,n}$ possess the P-F property and $Re(A) \leq Re(B)$, then

$$\rho(A) \leq \rho(B).$$

Moreover, if the matrices A and B^H possess the strong P-F property with $Re(A) \neq Re(B)$, then the inequality becomes strict.

Theorem (Type II)

If $A, B^H \in \mathbb{C}^{n,n}$ possess the complex P-F property with x and y the P-F eigenvectors, respectively, such that $Re(x) \geq |Im(x)|$, $Re(y) \geq |Im(y)|$ and $Re[(B-A)x] \geq |Im[(B-A)x]|$ or $Re[y^H(B-A)] \geq |Im[y^H(B-A)]|$. Then,

$$\rho(A) \leq \rho(B).$$

Moreover, if at least two of the above three inequalities are strict, then the inequality becomes is strict.

Bounds of spectral radius

Theorem (Type I)

Let $A^H \in \mathbb{C}^{n,n}$ possesses the P-F property, and let $x \geq 0$ $(x \neq 0)$ be such that $Re(A)x - \alpha x \geq 0$ for a constant $\alpha > 0$. Then,

$$\alpha \leq \rho(A).$$

Moreover, if $Re(A)x - \alpha x > 0$, then the inequality becomes strict.

Theorem (Type II)

Let $A^H \in \mathbb{C}^{n,n}$ possesses the complex P-F property, y be the P-F eigenvector with $Re(y) \geq |Im(y)|$ and $x \geq 0$ $(x \neq 0)$ be such that $Re(A)x - \alpha x \geq |Im(A)x|$ for a constant $\alpha > 0$ Then,

$$\alpha \leq \rho(A).$$

Moreover, if $Re(A)x - \alpha x > |Im(A)x|$, then the inequality becomes strict.

Bound of spectral radius

Theorem (Type I)

Let $A^H \in \mathbb{C}^{n,n}$ possesses the P-F property, and let $x > 0$ be such that $\alpha x - Re(A)x \geq 0$ for a constant $\alpha > 0$. Then,

$$\rho(A) \leq \alpha.$$

Moreover, if $\alpha x - Re(A)x > 0$, then the inequality becomes strict.

Theorem (Type II)

Let $A^H \in \mathbb{C}^{n,n}$ possesses the complex P-F property, let y be the Perron-Frobenius eigenvector with $Re(y) \geq |Im(y)|$, and let $x > 0$ be such that $\alpha x - Re(A)x \geq |Im(A)x|$ for a constant $\alpha > 0$. Then,

$$\rho(A) \leq \alpha.$$

Moreover, if $\alpha x - Re(A)x > |Im(A)x|$, then the inequality becomes strict.

Monotonicity properties

Theorem

Let $A \in \mathbb{C}^{n,n}$ possesses the (complex) P-F property with x the P-F eigenvector, and let $y \in \mathbb{C}^n$ be such that $y^H x > 0$. Then, the matrix

$$B = A + \epsilon x y^H, \quad \epsilon > 0,$$

possesses the (complex) P-F property, and there holds

$$\rho(A) < \rho(B).$$

Moreover, if A possesses the strong (complex) P-F property, then so does B.

Remark

Based on continuity arguments, we claim that the last result is valid also for $\hat{x}$ belonging to a cone of directions around x, while y is chosen such that $y^H \hat{x} > 0$.

Splittings of matrices

A splitting $A = M - N$ is called:

- M**-splitting if** M **is an** M**-matrix and** $N \geq 0$**,**
- **Regular splitting if** $M^{-1} \geq 0$ **and** $N \geq 0$**,**
- **Weak regular if** $M^{-1} \geq 0$ **and** $M^{-1}N \geq 0$**,**
- **Nonnegative if** $M^{-1}N \geq 0$**,**
- **Perron-Frobenius if** $M^{-1}N$ **possesses the P-F property,**
- **Complex Perron-Frobenius if** $M^{-1}N$ **possesses the complex P-F property.**

Theorem (Type I)

Let $A \in \mathbb{C}^{n,n}$ be a nonsingular matrix and the splitting $A = M - N$ be a P-F splitting, with x the P-F eigenvector. Then, the following properties are equivalent:

(1) $\rho(M^{-1}N) < 1$, (2) $A^{-1}N$ *possesses the P-F property,*
(3) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$, (4) $A^{-1}Mx \geq x$,
(5) $A^{-1}Nx \geq M^{-1}Nx$.

Theorem (Type II)

Let $A \in \mathbb{C}^{n,n}$ be a nonsingular matrix, and the splitting $A = M - N$ be a complex P-F splitting, with x the P-F eigenvector. Then, the following properties are equivalent:

(1) $\rho(M^{-1}N) < 1$, (2) $A^{-1}N$ *has the complex P-F property,*
(3) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$, (4) $Re(A^{-1}Mx) \geq Re(x)$,
(5) $Re(A^{-1}Nx) \geq Re(M^{-1}Nx)$.

Theorem (Type I)

Let $A \in \mathbb{C}^{n,n}$ be a nonsingular matrix, and let the splitting $A = M - N$ be a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector of $M^{-1}N$. If one of the following properties holds:
(i) There exists $y \in \mathbb{C}^n$ such that $Re(A^H y) \geq 0$, $Re(N^H y) \geq 0$ and $Re(y^H Ax) > 0$,
(ii) There exists $y \in \mathbb{C}^n$ such that $Re(A^H y) \geq 0$, $Re(M^H y) \geq 0$ and $Re(y^H Ax) > 0$, then,

$$\rho(M^{-1}N) < 1.$$

Theorem (Type II)

Let $A \in \mathbb{C}^{n,n}$ be a nonsingular matrix, and let the splitting $A = M - N$ be a complex Perron-Frobenius splitting of the first kind, with x the Perron-Frobenius eigenvector of $M^{-1}N$, such that $Re(x) \geq |Im(x)|$. If one of the following properties holds:
(i) There exists $y \in \mathbb{C}^n$, $Re(A^H y) \geq 0$, $Re(N^H y) \geq |Im(N^H y)|$ and $Re(y^H Ax) > 0$;
(ii) There exists $y \in \mathbb{C}^n$, $Re(A^H y) \geq 0$, $Re(M^H y) \geq |Im(M^H y)|$ and $Re(y^H Ax) > 0$ then,

$$\rho(M^{-1}N) < 1.$$

Theorem (Type I)

Let $A \in \mathbb{C}^{n,n}$ be a nonsingular matrix. If $A = M_1 - N_1$ and $A^H = M_2^H - N_2^H$ are two convergent Perron-Frobenius splittings of the first kind and of the second kind, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^H := (M_2^{-1}N_2)^H$, and with $x \geq 0$, $y \geq 0$, the associated Perron-Frobenius eigenvectors, respectively, such that

$$Re(y^H A^{-1}) \geq |Im(y^H A^{-1})|, \quad y^H x > 0,$$

$$Re(N_2 x) - Re(N_1 x) \geq |Im(N_2 x) - Im(N_1 x)|,$$

then

$$\rho(T_1) \leq \rho(T_2).$$

Moreover, if $Re(y^H A^{-1}) > |Im(y^H A^{-1})|$ and $Re(N_2 x) - Re(N_1 x) > |Im(N_2 x) - Im(N_1 x)|$, then

$$\rho(T_1) < \rho(T_2).$$

Theorem (Type II)

Let $A \in \mathbb{C}^{n,n}$ be a nonsingular matrix. If $A = M_1 - N_1$ and $A^H = M_2^H - N_2^H$ are two convergent complex Perron-Frobenius splittings of the first kind and of the second kind, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^H := (M_2^{-1}N_2)^H$, and with x, y the associated Perron-Frobenius eigenvectors, respectively, such that

$$Re(y^H A^{-1}) \geq |Im(y^H A^{-1})|, \quad Re(y^H x) > 0,$$

$$Re(N_2 x) - Re(N_1 x) \geq |Im(N_2 x) - Im(N_1 x)|,$$

then

$$\rho(T_1) \leq \rho(T_2).$$

Moreover, if $Re(y^H A^{-1}) > |Im(y^H A^{-1})|$ and if $Re(N_2 x) - Re(N_1 x) > |Im(N_2 x) - Im(N_1 x)|$, then

$$\rho(T_1) < \rho(T_2).$$

Perron-Frobenius splittings

Theorem (Type I)

Let $A \in \mathbb{C}^{n,n}$ be a nonsingular matrix. If $A = M_1 - N_1$ and $A^H = M_2^H - N_2^H$ are two convergent Perron-Frobenius splittings of the first kind and of the second kind, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^H := (M_2^{-1}N_2)^H$, and with $x \geq 0$, $y \geq 0$ the associated Perron-Frobenius eigenvectors, respectively, $Re(N_1x) \geq |Im(N_1x)|$ and $Re(y^H(M_1^{-1} - M_2^{-1})) \geq |Im(y^H(M_1^{-1} - M_2^{-1}))|$, $y^Hx > 0$, then

$$\rho(T_1) \leq \rho(T_2).$$

Moreover, if $Re(y^H(M_1^{-1} - M_2^{-1})) > |Im(y^H(M_1^{-1} - M_2^{-1}))|$ and $N_1x \neq 0$, then the inequality becomes strict, while if $y^HM_1^{-1} = y^HM_2^{-1}$, then the inequality becomes an equality.

Perron-Frobenius splittings

Theorem (Type II)

Let $A \in \mathbb{C}^{n,n}$ be a nonsingular matrix. If $A = M_1 - N_1$ and $A^T = M_2^H - N_2^H$ are two convergent complex Perron-Frobenius splittings of the first kind and of the second kind, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^H := (M_2^{-1}N_2)^H$, and if x, y are the associated Perron-Frobenius eigenvectors, respectively, with
$Re(N_1x) \geq |Im(N_1x)|$ and
$Re(y^H(M_1^{-1} - M_2^{-1})) \geq |Im(y^H(M_1^{-1} - M_2^{-1}))|$, $Re(y^Hx) > 0$, then

$$\rho(T_1) \leq \rho(T_2).$$

Moreover, if $Re(y^H(M_1^{-1} - M_2^{-1})) > |Im(y^H(M_1^{-1} - M_2^{-1}))|$ and if $N_1x \neq 0$, then the inequality becomes strict, while if $y^HM_1^{-1} = y^HM_2^{-1}$, then the inequality becomes an equality.

Thank You

for your

Attention!!!

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