Max algebraic powers of nonnegative matrices

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INTRODUCTION

General

- Consider nonnegative numbers $\mathbb{R}_+$ equipped with

$$ab,\ \ a \oplus b := \max(a, b).$$

- Extension to matrices and vectors: For $A = (a_{ij}),\ B = (b_{ij})$:

$$(A \oplus B)_{ij} := (a_{ij} \oplus b_{ij})$$

$$(A \otimes B)_{ij} := \bigoplus_k a_{ik} \otimes b_{kj}$$

$$A^k := \overbrace{A \otimes \cdots \otimes A}^{k}.$$

Motivation: cyclicity theorem

Theorem (Cyclicity Theorem)

Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible and $\lambda(A) = 1$, then

- *$A, A^2, A^3 \ldots$ is ultimately periodic:*
  $\exists T(A)\ \exists\gamma$ such that $A^{r+\gamma} = A^r$ for $r \geq T(A)$.

$\sim$ Cohen, Dubois, Quadrat and Viot, 1984.

- It follows that the **orbits** $\{A^l x\}$ are ultimately periodic.

This talk

We study:

1. **Attraction cones** $\{x\colon A^l x \text{ has ultimate period } 1\}$: describe by system of equations.

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We study:

1. **Attraction cones** $\{x \colon A^l x \text{ has ultimate period } 1\}$: describe by system of equations.
2. **CSR expansions:** generalization of Cyclicity Theorem to the **reducible case**.

Literature

1. P. Butkovič, Max-linear systems: theory and applications, 2010 (to appear).

2. S. Sergeev, Cyclic classes and attraction cones in max algebra, arXiv:0903.3960.

3. S. Sergeev, H. Schneider, CSR expansions of matrix powers in max algebra, arXiv:0912.2534.

PRELIMINARIES

Powers and paths

- Let $A \in \mathbb{R}_+^{n\times n}$. For a path $P = i_1 \to \cdots \to i_k$, denote the **weight** and the **length** of P by

$$w(P) := a_{i_1 i_2} \cdot \ldots \cdot a_{i_{k-1} i_k},\ \ l(P) = k-1.$$

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- **Path sense** of matrix powers $A^l = (a_{ij}^{(l)})$:

$$a_{ij}^{(l)} = \max\{w(P) \colon i \overset{P}{\longrightarrow} j, \;\; l(P) = l\}.$$

Kleene star

- **Kleene star** is analogue of $(I - A)^{-1}$

$$A^* := I \oplus A \oplus A^2 \oplus \cdots$$

- **Converges** $\Leftrightarrow$ no cycles P with $w(P) > 1$.

  In this case, $A^* = I \oplus A \oplus \cdots \oplus A^{n-1}$.

Kleene star

- a^*_{ij} is the **maximal weight of paths** connecting i to j (for $i \neq j$).

Maximal cycle geometric mean

- The **maximum cycle geometric mean** of A is

$$\lambda(A) := \max_{\text{cycles } P} \sqrt[l(P)]{w(P)}.$$

- $\lambda(\alpha A) = \alpha\lambda(A) \Rightarrow$ any matrix with $\lambda(A) \neq 0$ can be "normalised":

$$\lambda(A/\lambda(A)) = 1.$$

Critical graph

- Cycles P where $\lambda(A)$ is attained are called **critical**:

$$\sqrt[l(P)]{w(P)} = \lambda(A).$$

- The **critical graph** of A, denoted $\mathcal{G}^c(A)$, consists of all nodes and edges which belong to critical cycles.

Cyclicity of critical graphs

- $\mathcal{G}^c(A)$ has n_c strongly connected components (s.c.c.) $\mathcal{G}^c_\mu$, for $\mu = 1, \ldots, n_c$.
- The cyclicity of $\mathcal{G}^c_\mu$, denoted by γ_μ, is the g.c.d. of the lengths of all simple cycles.
- The cyclicity of $\mathcal{G}^c(A)$, denoted by γ, is the l.c.m. of all γ_μ, for $\mu = 1, \ldots, n_c$.

Critical matrix

- $A^{[C]} = (a_{ij}^{[C]}) \in \{1, 0\}^{n \times n}$:

$$a_{ij}^{[C]} := \begin{cases} 1, & \text{if } (i,j) \in \mathcal{G}^c(A), \\ 0, & \text{if } (i,j) \notin \mathcal{G}^c(A). \end{cases}$$

- $(A^k)^{[C]} = (A^{[C]})^k$.

Critical matrix (strongly connected case)

- Generally $(A^{[C]})^{k+\gamma} = (A^{[C]})^k$ after $k \sim n^2/\gamma + \gamma$.

  $\sim$ Š. Schwartz, On a sharp estimation in the theory of binary relations on a finite set, 1970.

Critical matrix (strongly connected case)

- If $\gamma = 1$ then $(A^{[C]})^k$ becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

after $(n-1)^2 + 1$.

$\sim$ H. Wielandt, Unzerlegbare nichtnegative Matrizen, 1950.

Visualization

- **Idea:** apply diagonal similarity scaling $D^{-1}AD$ to **visualize** the critical matrix.

Visualization

- For any $A \in \mathbb{R}_{+}^{n\times n}$, there exist diagonal D s.t. $B = D^{-1}AD$ satisfies

$$b_{ij} \leq \lambda(A) \quad \forall i,j.$$
$$b_{ij} = \lambda(A) \quad \forall (i,j) \in \mathcal{G}^c(A) = \mathcal{G}^c(B).$$

  $\sim$ M. Fiedler and V. Pták, *Diagonally dominant matrices* (Czechoslovak Math. J.), 1967.

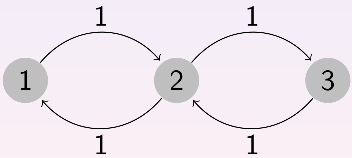
- Such matrix will be called visualized.

Example

- **Visualized matrix and its critical graph**

$$A = \begin{pmatrix} 0.5 & 1 & 0.5 \\ 1 & 0.5 & 1 \\ 0.5 & 1 & 0.5 \end{pmatrix},$$

$\mathcal{G}^c(A)$:



CYCLICITY THEOREM: IRREDUCIBLE CASE

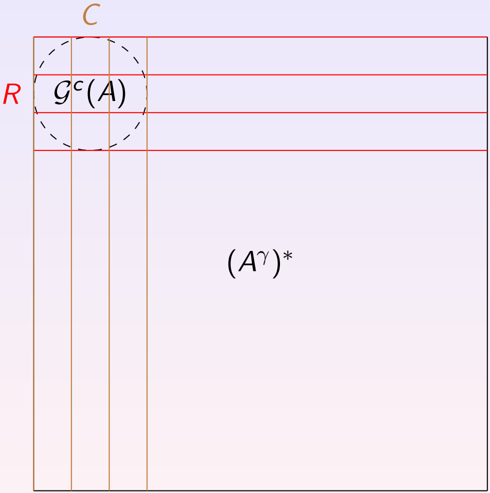
CSR-terms

- Going to express the **cyclicity theorem** in terms of *CSR*-representation...

CSR-terms

- Assume $\lambda(A) = 1$ and A is visualized.
- Let C, resp. R be the matrices extracted from the critical columns, resp. rows, of $(A^{\gamma})^*$.
- Let $S := A^{[C]}$ (Boolean matrix).

C and R terms



Cyclicity Theorem

- New *CSR* formulation:

Theorem (Cyclicity Theorem)

Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible and visualized, $\lambda(A) = 1$, then there exists $T(A)$ such that $A^l = CS^lR$ for all $l \geq T(A)$.

In particular:

- $A^{l+\gamma} = A^l$ for all $l \geq T(A)$.
- S^lR yields critical rows of A^l, $l \geq T(A)$.
- CS^l yields critical columns of A^l, $l \geq T(A)$.

SYSTEM FOR ATTRACTION CONE

Attraction cones

- **Attraction cone:**

$$\mathrm{Attr}(A) = \{x\colon A^l x \text{ has ultimate period } 1\}.$$

- More formally,

$$\mathrm{Attr}(A) := \{x\colon A^l x = A^{l+1} x \; \forall l \geq T(A)\}.$$

Attraction cones

- Using *CSR*, we obtain

$$\mathrm{Attr}(A) := \{x \colon Rx = SRx\}.$$

- $Rx = SRx$:
  expressions on the l.h.s. = *permuted* expressions on the r.h.s.

Attraction cones

- $Rx = SRx$ consists of several chains $\Leftrightarrow$ components of $\mathcal{G}^c(A)$.

Cancellation law

- Cancellation law:

$$ax \oplus b = cx \oplus d, \;\; a < c \;\; \Leftrightarrow \;\; b = cx \oplus d,$$

Example

- **Example:** Take a visualized matrix

$$A = \begin{pmatrix} 0.6 & 1 & 0.7 & 0.1 & 0.7 & 0.5 \\ 0.3 & 0.7 & 1 & 0.4 & 0.2 & 0.1 \\ 1 & 0.1 & 0.2 & 0.9 & 0.5 & 0.1 \\ 0.1 & 0.4 & 0.6 & 0.3 & 1 & 0.2 \\ 0.4 & 0.4 & 0.6 & 0.5 & 0.8 & 1 \\ 0.3 & 0.6 & 0.1 & 1 & 0.9 & 0.2 \end{pmatrix}$$

- $\mathcal{G}^c(A)$ consists of cycles (123) and (456).

Example

- The critical matrix $S = A^{[C]}$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- $\mathcal{G}^c(A)$ consists of cycles (123) and (456).

Example

- Matrices R and C are precisely

$$A^9 = \begin{pmatrix} 1 & 0.7 & 0.7 & 0.9 & 0.81 & 0.73 \\ 0.7 & 1 & 0.7 & 0.73 & 0.9 & 0.81 \\ 0.7 & 0.7 & 1 & 0.81 & 0.73 & 0.9 \\ 0.6 & 0.6 & 0.54 & 1 & 0.9 & 0.81 \\ 0.54 & 0.6 & 0.6 & 0.81 & 1 & 0.9 \\ 0.6 & 0.54 & 0.6 & 0.9 & 0.81 & 1 \end{pmatrix}$$

Example

- Take $A^{10} = CSR$:

$$A^{10} = \begin{pmatrix} 0.7 & 1 & 0.7 & 0.73 & 0.9 & 0.81 \\ 0.7 & 0.7 & 1 & 0.81 & 0.73 & 0.9 \\ 1 & 0.7 & 0.7 & 0.9 & 0.81 & 0.73 \\ 0.54 & 0.6 & 0.6 & 0.81 & 1 & 0.9 \\ 0.6 & 0.54 & 0.6 & 0.9 & 0.81 & 1 \\ 0.6 & 0.6 & 0.54 & 1 & 0.9 & 0.81 \end{pmatrix}$$

- obtained from A^9 by permuting rows (123) and (456).

Example

- Take $A^{11} = CS^2R$:

$$A^{11} = \begin{pmatrix} 0.7 & 0.7 & 1 & 0.81 & 0.73 & 0.9 \\ 1 & 0.7 & 0.7 & 0.9 & 0.81 & 0.73 \\ 0.7 & 1 & 0.7 & 0.73 & 0.9 & 0.81 \\ 0.6 & 0.54 & 0.6 & 0.9 & 0.81 & 1 \\ 0.6 & 0.6 & 0.54 & 1 & 0.9 & 0.81 \\ 0.54 & 0.6 & 0.6 & 0.81 & 1 & 0.9 \end{pmatrix}$$

- obtained from A^{10} by permuting rows (123) and (456).

Example

- System for attraction cone:
  two chains $\Leftrightarrow \{1,2,3\}, \{4,5,6\}$.

$$A^{11}_{1\cdot} \otimes x = A^{11}_{2\cdot} \otimes x = A^{11}_{3\cdot} \otimes x,$$
$$A^{11}_{4\cdot} \otimes x = A^{11}_{5\cdot} \otimes x = A^{11}_{6\cdot} \otimes x.$$

- After cancellation few terms survive:

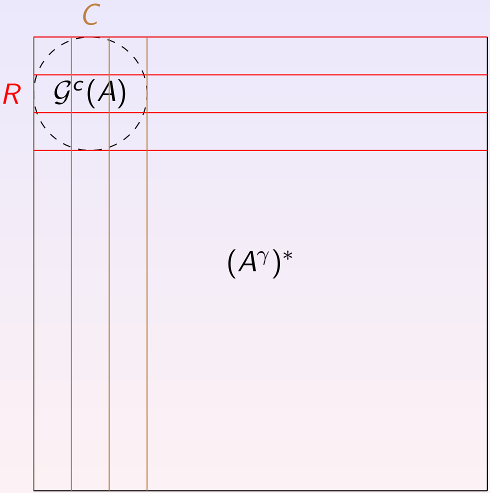
$$x_3 \oplus 0.9x_6 = x_1 \oplus 0.9x_4 = x_2 \oplus 0.9x_5,$$
$$0.6x_1 \oplus 0.6x_3 \oplus x_6 = 0.6x_1 \oplus 0.6x_2 \oplus x_4 =$$
$$= 0.6x_2 \oplus 0.6x_3 \oplus x_5.$$

REDUCIBLE CASE

CSR terms

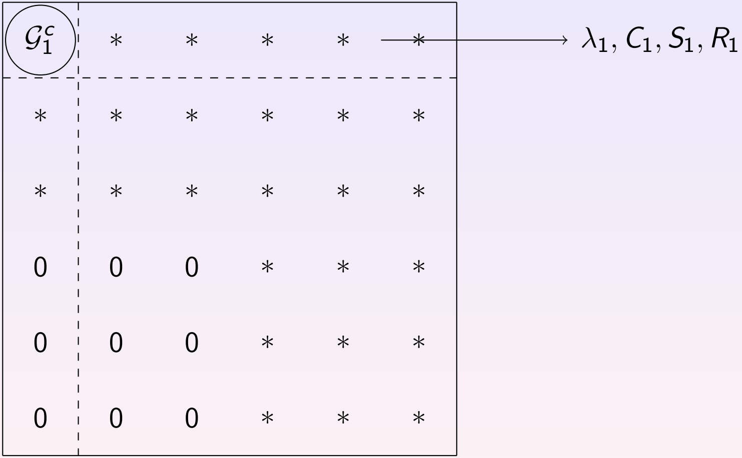
- C, S and R can be defined as before:
  C and R come from $(A^{\gamma})^*$, and $S = A^{[C]}$.
- We do not hope for $A^l = CS^lR$.
- We can approximate A^l by $CS^lR \Rightarrow$
  CSR expansions !

C and R terms

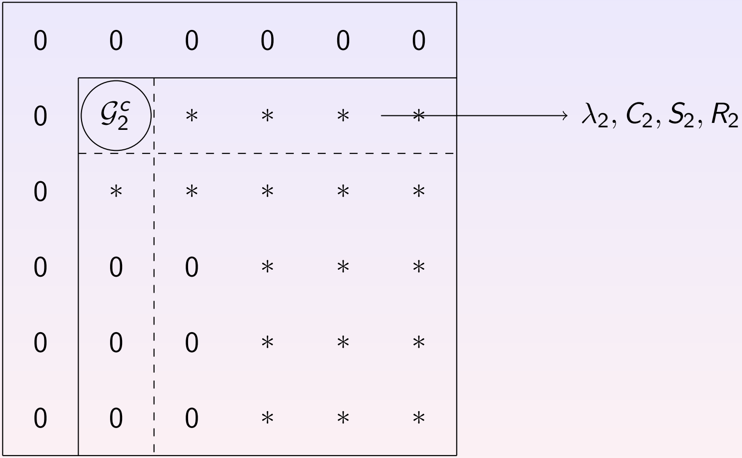


CSR EXPANSIONS: I

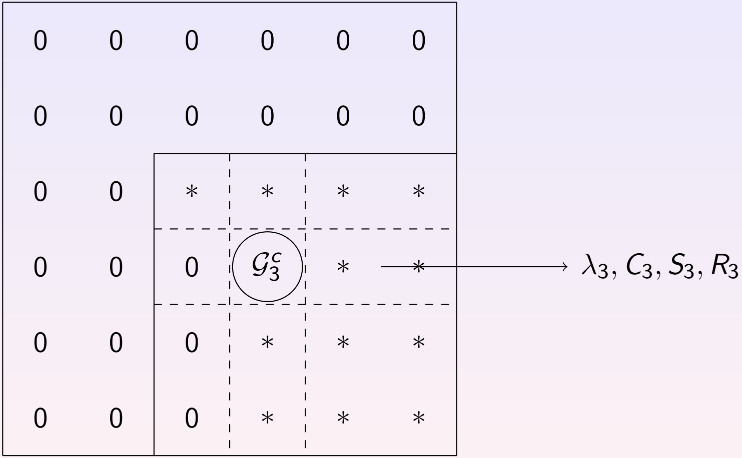
Nachtigall expansion



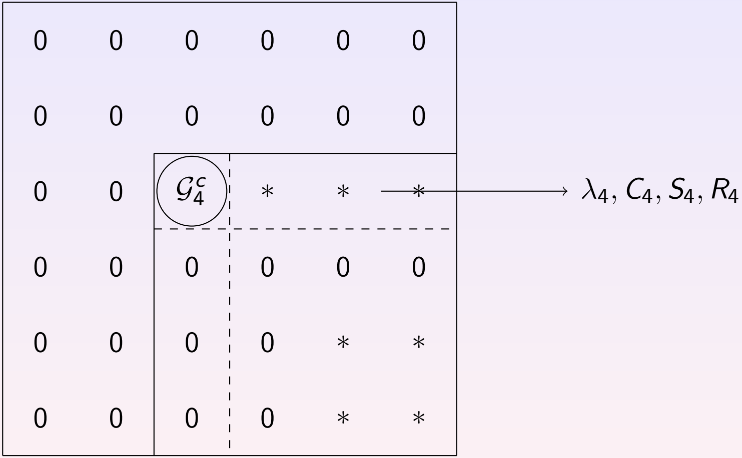
Nachtigall expansion



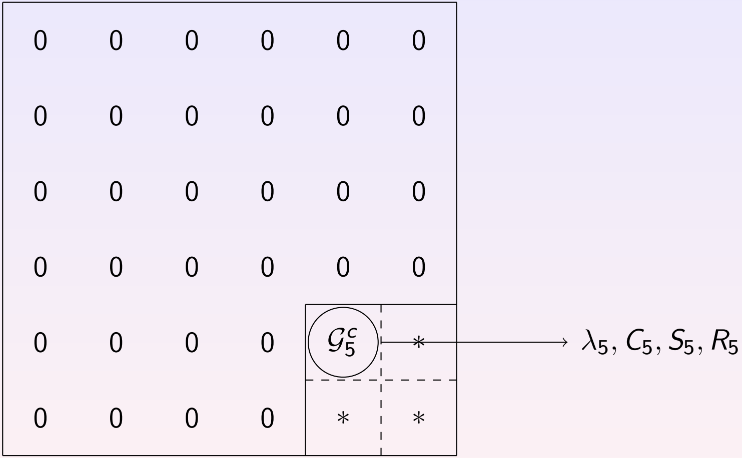
Nachtigall expansion



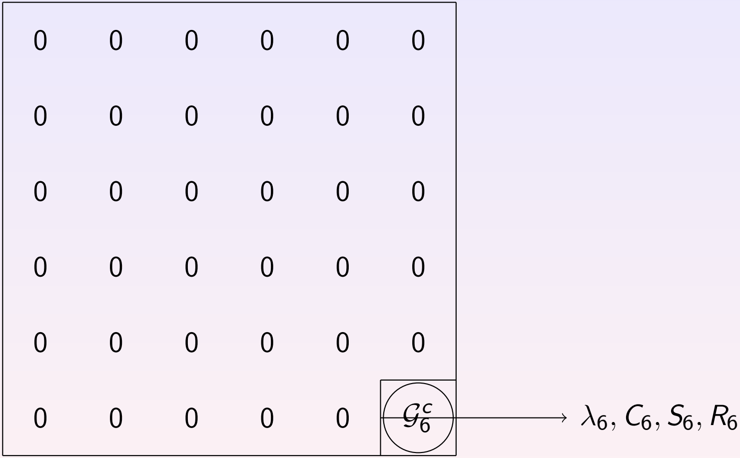
Nachtigall expansion



Nachtigall expansion



Nachtigall expansion



Nachtigall expansion

Theorem

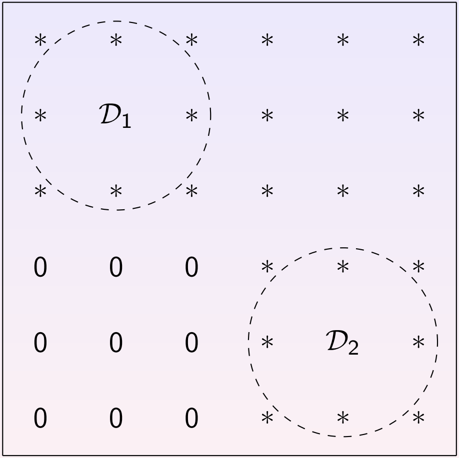
For all $l \geq 3n^2$

$$A^l = \bigoplus_{\mu} \lambda_\mu^l C_\mu S_\mu^l R_\mu.$$

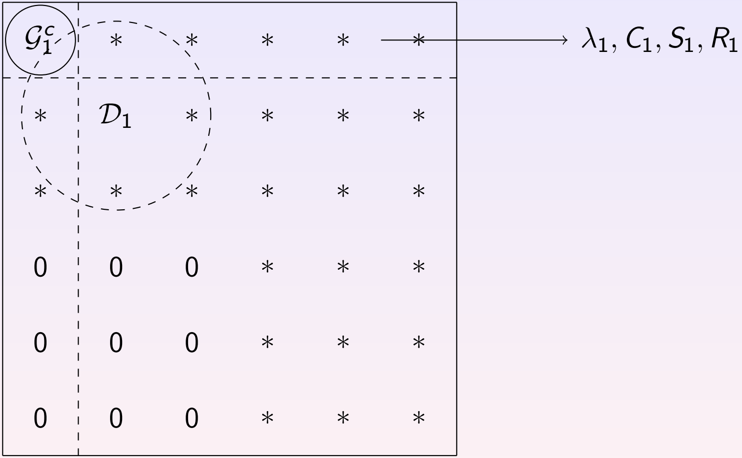
- Yet not the generalization of the cyclicity theorem...

CSR EXPANSIONS: II

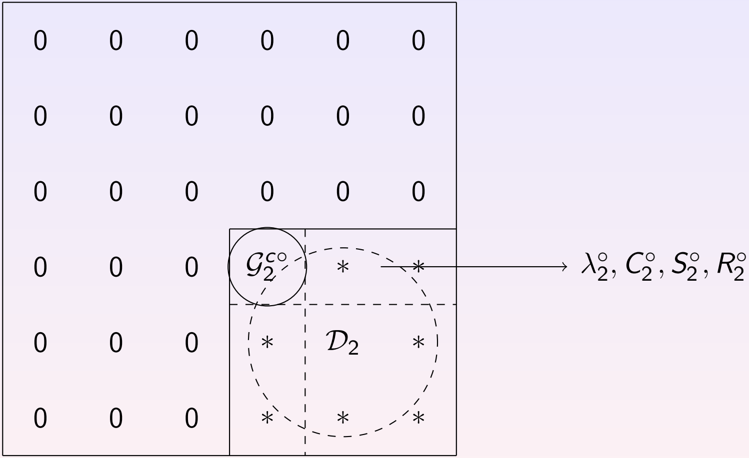
Ultimate expansion



Ultimate expansion



Ultimate expansion



Ultimate expansion

Theorem

There is an integer $T(A)$ such that for all $l \geq T(A)$

$$A^l = \bigoplus_{\mu} (\lambda_{\mu}^{\circ})^l C_{\mu}^{\circ} (S_{\mu}^{\circ})^l R_{\mu}^{\circ}.$$

- We recover cyclicity theorem if A **irreducible**.

CSR expansions

- **Idea:** Expansion II is the **limit case** of Expansion I at $l \geq T(A)$.

Conclusions

- We obtained *CSR* form of the Cyclicity Theorem,
- We described $\mathrm{Attr}(A) = \{x\colon Rx = SRx\}$,
- We obtained *CSR* expansions I and II (II is a generalization of Cyclicity Theorem).

Further research

- in the **reducible case**, describe well-behaved matrices (e.g. such that $\{A^t x\}$ is ultimately periodic for all x).
- in the **reducible case**, describe initial vectors with well-behaved orbits (e.g. attraction cones in the reducible case).

- Wielandt-type bounds in max algebra (like $(n-1)^2+1$),
- Semigroups generated by several matrices

THANK YOU!