Max-Linear Systems: Feasibility and Reachability

P.Butkovic
University of Birmingham

CONTENTS

CONTENTS

CONTENTS

CONTENTS

CONTENTS

CONTENTS

- $$\begin{aligned} a \oplus b &= \max(a, b) \\ a \otimes b &= a + b \\ a, b &\in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \end{aligned}$$

- $$\begin{aligned} a \oplus b &= \max(a, b) \\ a \otimes b &= a + b \\ a, b &\in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \end{aligned}$$

- $$\begin{aligned} a \oplus b &= \max(a, b) \\ a \otimes b &= ab \\ a, b &\in \mathbb{R}_+ \end{aligned}$$

- $$\begin{aligned} a \oplus b &= \max(a, b) \\ a \otimes b &= a + b \\ a, b &\in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \end{aligned}$$

- $$\begin{aligned} a \oplus b &= \max(a, b) \\ a \otimes b &= ab \\ a, b &\in \mathbb{R}_+ \end{aligned}$$

- $$\text{Isomorphism} : x \longrightarrow 2^x$$

- $$\begin{aligned} a \oplus b &= \max(a,b) \\ a \otimes b &= a+b \\ a,b &\in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \end{aligned}$$

- $$\begin{aligned} a \oplus b &= \max(a,b) \\ a \otimes b &= ab \\ a,b &\in \mathbb{R}_+ \end{aligned}$$

- $$\text{Isomorphism} : x \longrightarrow 2^x$$

- $$\text{finite} \leftrightarrow \text{positive}$$

Basic properties ($\varepsilon = -\infty$):

$$a \oplus b = b \oplus a$$
$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$
$$a \oplus \varepsilon = a = \varepsilon \oplus a$$

$$a \otimes b = b \otimes a$$
$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$
$$a \otimes \varepsilon = \varepsilon = \varepsilon \otimes a$$
$$a \otimes 0 = a = 0 \otimes a$$

$$(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c$$
$$a \oplus b = a \text{ or } b$$

Extension to matrices and vectors:

$$A \oplus B = \left(a_{ij} \oplus b_{ij}\right)$$
$$A \otimes B = \left(\sum_k^{\oplus} a_{ik} \otimes b_{kj}\right)$$
$$\alpha \otimes A = \left(\alpha \otimes a_{ij}\right)$$

$$A \oplus B = B \oplus A$$
$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$
$$A \otimes \varepsilon = \varepsilon = \varepsilon \otimes A$$

$$[\text{not } A \otimes B = B \otimes A]$$
$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$
$$A \otimes I = A = I \otimes A$$

$$(A \oplus B) \otimes C = A \otimes C \oplus B \otimes C$$
$$A \otimes (B \oplus C) = A \otimes B \oplus A \otimes C$$

- A^{-1} exists $\Longleftrightarrow$ A is a generalised permutation matrix

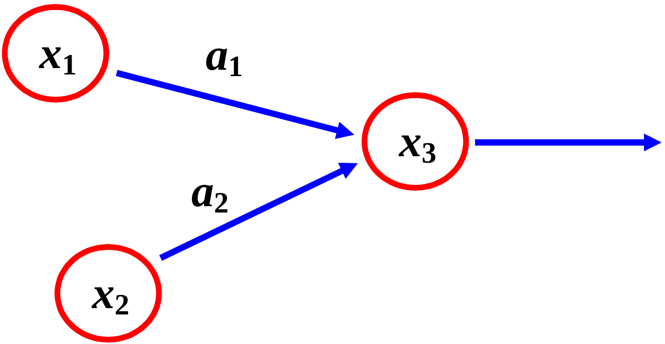
- A^{-1} exists $\Longleftrightarrow$ A is a generalised permutation matrix
- $\oplus$ idempotent, not invertible

- A^{-1} exists $\Longleftrightarrow$ A is a generalised permutation matrix
- $\bigoplus$ idempotent, not invertible
- $a \oplus a = a$

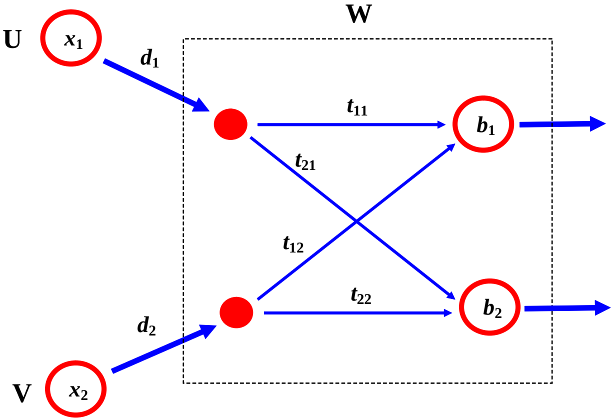
- A^{-1} exists $\Longleftrightarrow$ A is a generalised permutation matrix
- $\oplus$ idempotent, not invertible
- $a \oplus a = a$
- $(a \oplus b)^k = a^k \oplus b^k$

- A^{-1} exists $\Longleftrightarrow$ A is a generalised permutation matrix
- $\oplus$ idempotent, not invertible
- $a \oplus a = a$
- $(a \oplus b)^k = a^k \oplus b^k$
- $(A \oplus B)^k \neq A^k \oplus B^k$

- A^{-1} exists $\Longleftrightarrow$ A is a generalised permutation matrix
- $\oplus$ idempotent, not invertible
- $a \oplus a = a$
- $(a \oplus b)^k = a^k \oplus b^k$
- $(A \oplus B)^k \neq A^k \oplus B^k$
- $(I \oplus A)^k = I \oplus A \oplus A^2 \oplus ... \oplus A^k$



$$
\begin{aligned}
x_3 &= \max\left(x_1 + a_1, x_2 + a_2\right) \\
&= a_1 \otimes x_1 \oplus a_2 \otimes x_2 \\
&= \left(a_1, a_2\right) \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a^T \otimes x
\end{aligned}
$$



$$\left.\begin{aligned} b_1 &= \max\left(x_1 + d_1 + t_{11}, x_2 + d_2 + t_{12}\right) \\ b_2 &= \max\left(x_1 + d_1 + t_{21}, x_2 + d_2 + t_{22}\right) \end{aligned}\right\} b = A \otimes x$$

- Machines $M_1, ..., M_n$ produce parts for products $P_1, ..., P_m$

- Machines $M_1, ..., M_n$ produce parts for products $P_1, ..., P_m$
- x_j . . . starting time of machine M_j

- Machines $M_1, ..., M_n$ produce parts for products $P_1, ..., P_m$
- x_j ... starting time of machine M_j
- a_{ij} ... time M_j needs to prepare the component for P_i

- Machines $M_1, ..., M_n$ produce parts for products $P_1, ..., P_m$
- x_j ... starting time of machine M_j
- a_{ij} ... time M_j needs to prepare the component for P_i
- All parts for P_i will be ready at time $\max\left(x_1 + a_{i1}, \ldots, x_n + a_{in}\right)$

- Machines $M_1, ..., M_n$ produce parts for products $P_1, ..., P_m$
- x_j . . . starting time of machine M_j
- a_{ij} . . . time M_j needs to prepare the component for P_i
- All parts for P_i will be ready at time $\max\left(x_1 + a_{i1}, \ldots, x_n + a_{in}\right)$
- $b_1, ..., b_m$ ... **required completion times** for products $P_1, ..., P_m$

- Machines $M_1, \ldots, M_n$ produce parts for products $P_1, \ldots, P_m$
- x_j ... starting time of machine M_j
- a_{ij} ... time M_j needs to prepare the component for P_i
- All parts for P_i will be ready at time $\max\left(x_1 + a_{i1}, \ldots, x_n + a_{in}\right)$
- $b_1, \ldots, b_m$ ... **required completion times** for products $P_1, \ldots, P_m$
- Then the starting times should satisfy $\max\left(x_1 + a_{i1}, \ldots, x_n + a_{in}\right) = b_i \ (i = 1, \ldots, m)$

- Machines $M_1, ..., M_n$ produce parts for products $P_1, ..., P_m$
- x_j ... starting time of machine M_j
- a_{ij} ... time M_j needs to prepare the component for P_i
- All parts for P_i will be ready at time $\max\left(x_1 + a_{i1}, \ldots, x_n + a_{in}\right)$
- $b_1, ..., b_m$ ... **required completion times** for products $P_1, ..., P_m$
- Then the starting times should satisfy $\max\left(x_1 + a_{i1}, \ldots, x_n + a_{in}\right) = b_i \ (i = 1, ..., m)$
- In max-notation: $\sum_j^{\oplus} a_{ij} \otimes x_j = b_i \quad (i = 1, \ldots, m)$

- Machines $M_1, ..., M_n$ produce parts for products $P_1, ..., P_m$
- x_j ... starting time of machine M_j
- a_{ij} ... time M_j needs to prepare the component for P_i
- All parts for P_i will be ready at time $\max\left(x_1 + a_{i1}, \ldots, x_n + a_{in}\right)$
- $b_1, ..., b_m$ ... **required completion times** for products $P_1, ..., P_m$
- Then the starting times should satisfy $\max\left(x_1 + a_{i1}, \ldots, x_n + a_{in}\right) = b_i$ $(i = 1, ..., m)$
- In max-notation: $\sum_j^{\oplus} a_{ij} \otimes x_j = b_i \quad (i = 1, \ldots, m)$
- Equivalently: $A \otimes x = b$

- Machines $M_1, ..., M_n$ produce parts for products $P_1, ..., P_m$
- x_j ... starting time of machine M_j
- a_{ij} ... time M_j needs to prepare the component for P_i
- All parts for P_i will be ready at time $\max(x_1 + a_{i1}, \ldots, x_n + a_{in})$
- $b_1, ..., b_m$ ... **required completion times** for products $P_1, ..., P_m$
- Then the starting times should satisfy $\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = b_i \ (i = 1, ..., m)$
- In max-notation: $\sum_j^{\oplus} a_{ij} \otimes x_j = b_i \quad (i = 1, \ldots, m)$
- Equivalently: $A \otimes x = b$
- $A \otimes x$ ... vector of **actual completion times**

- Machines $M_1, \ldots, M_n$ produce parts for products $P_1, \ldots, P_m$
- x_j . . . starting time of machine M_j
- a_{ij} . . . time M_j needs to prepare the component for P_i
- All parts for P_i will be ready at time $\max(x_1 + a_{i1}, \ldots, x_n + a_{in})$
- $b_1, \ldots, b_m$ ... **required completion times** for products $P_1, \ldots, P_m$
- Then the starting times should satisfy $\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = b_i \ (i = 1, \ldots, m)$
- In max-notation: $\sum_j^{\oplus} a_{ij} \otimes x_j = b_i \quad (i = 1, \ldots, m)$
- Equivalently: $A \otimes x = b$
- $A \otimes x$ ... vector of **actual completion times**
- If no solution then $A \otimes x \leq b$ but as tight as possible

- A ... Production/transportation matrix

- A ... Production/transportation matrix
- b ... Vector of required completion times

- A ... Production/transportation matrix
- b ... Vector of required completion times
- x ... Vector of starting times

- A ... Production/transportation matrix
- b ... Vector of required completion times
- x ... Vector of starting times
- Given A and b, find x so that

$$A \otimes x \leq b$$

is satisfied as tightly as possible

- A ... Production/transportation matrix
- b ... Vector of required completion times
- x ... Vector of starting times
- Given A and b, find x so that

$$A \otimes x \leq b$$

  is satisfied as tightly as possible
- Let $\overline{x} = A^* \otimes' b$, where $A^* = -A^T$ and $\otimes'$ is in min-algebra

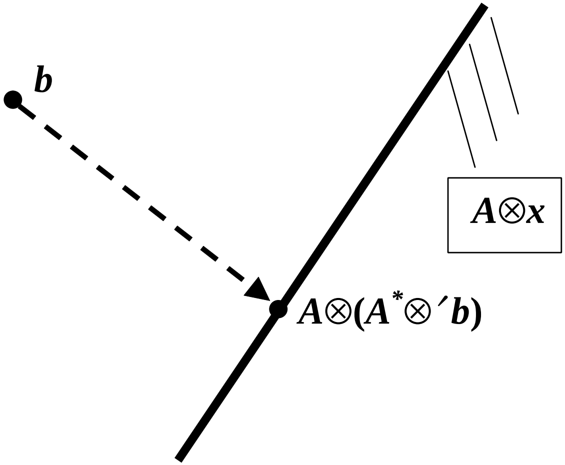
- A ... Production/transportation matrix
- b ... Vector of required completion times
- x ... Vector of starting times
- Given A and b, find x so that

$$A \otimes x \leq b$$

is satisfied as tightly as possible

- Let $\overline{x} = A^{*} \otimes' b$, where $A^{*} = -A^{T}$ and $\otimes'$ is in min-algebra
- Then $A \otimes \overline{x} \leq b$ and $A \otimes \overline{x}$ is the best Chebyshev approximation of b

Solving $A \otimes x = b$



- If two such processes are given:

- If two such processes are given:
- $A \otimes x = b$

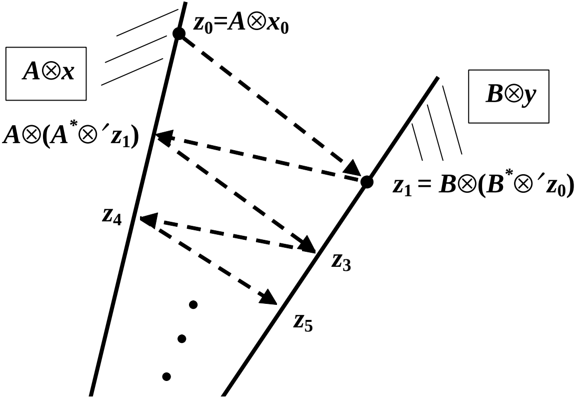
- If two such processes are given:
- $A \otimes x = b$
- $B \otimes y = c$

- If two such processes are given:
- $A \otimes x = b$
- $B \otimes y = c$
- ... and they have to be synchronised $(b = c)$:

- If two such processes are given:
- $A \otimes x = b$
- $B \otimes y = c$
- ... and they have to be synchronised $(b = c)$:
- $A \otimes x = B \otimes y$

- If two such processes are given:
- $A \otimes x = b$
- $B \otimes y = c$
- ... and they have to be synchronised ($b = c$):
- $A \otimes x = B \otimes y$
- Alternating Method

Solving $A \otimes x = B \otimes y$



- Find x so that $A \otimes x \leq b$ is satisfied as tightly as possible ($feasibility$) ... $O\left(n^2\right)$

- Find x so that $A \otimes x \leq b$ is satisfied as tightly as possible (*feasibility*) ... $O\left(n^2\right)$
  - Find x so that the range norm of $A \otimes x$ is min/max ... $O\left(n^2\right)$

- Find x so that $A \otimes x \leq b$ is satisfied as tightly as possible (*feasibility*) ... $O\left(n^2\right)$
  - Find x so that the range norm of $A \otimes x$ is min/max ... $O\left(n^2\right)$
  - Find π so that $A \otimes x \leq b(\pi)$ can be satisfied as tightly as possible... NP-complete

- Find x so that $A \otimes x \leq b$ is satisfied as tightly as possible (*feasibility*) ... $O\left(n^2\right)$
  - Find x so that the range norm of $A \otimes x$ is min/max ... $O\left(n^2\right)$
  - Find π so that $A \otimes x \leq b(\pi)$ can be satisfied as tightly as possible... NP-complete
- $A \otimes x = B \otimes y$ (*synchronisation of two processes*) ... Alternating Method (pseudopolynomial)

- Find x so that $A \otimes x \leq b$ is satisfied as tightly as possible (*feasibility*) ... $O\left(n^2\right)$
  - Find x so that the range norm of $A \otimes x$ is min/max ... $O\left(n^2\right)$
  - Find π so that $A \otimes x \leq b(\pi)$ can be satisfied as tightly as possible... NP-complete
- $A \otimes x = B \otimes y$ (*synchronisation of two processes*) ... Alternating Method (pseudopolynomial)
- $A \otimes x = B \otimes x$ ... can be transformed to $A \otimes x = B \otimes y$

- Find x so that $A \otimes x \leq b$ is satisfied as tightly as possible ($\textit{feasibility}$) ... $O\left(n^2\right)$
  - Find x so that the range norm of $A \otimes x$ is min/max ... $O\left(n^2\right)$
  - Find π so that $A \otimes x \leq b(\pi)$ can be satisfied as tightly as possible... NP-complete
- $A \otimes x = B \otimes y$ (*synchronisation of two processes*) ... Alternating Method (pseudopolynomial)
- $A \otimes x = B \otimes x$ ... can be transformed to $A \otimes x = B \otimes y$
- $A \otimes x = \lambda \otimes B \otimes x$ (*linked synchronisation*, $x = \lambda \otimes y$)

- Find x so that $A \otimes x \leq b$ is satisfied as tightly as possible (*feasibility*) ... $O\left(n^2\right)$
  - Find x so that the range norm of $A \otimes x$ is min/max ... $O\left(n^2\right)$
  - Find π so that $A \otimes x \leq b(\pi)$ can be satisfied as tightly as possible... NP-complete
- $A \otimes x = B \otimes y$ (*synchronisation of two processes*) ... Alternating Method (pseudopolynomial)
- $A \otimes x = B \otimes x$ ... can be transformed to $A \otimes x = B \otimes y$
- $A \otimes x = \lambda \otimes B \otimes x$ (*linked synchronisation*, $x = \lambda \otimes y$)
- One-sided problems easier than in LA, two-sided harder

- Machines $M_1, \ldots, M_n$ work interactively and in stages

- Machines $M_1, ..., M_n$ work interactively and in stages
- $x_i(r)$ ... starting time of the r^{th} stage on machine M_i $(i = 1, \ldots, n; r = 0, 1, ...)$

- Machines $M_1, ..., M_n$ work interactively and in stages
- $x_i(r)$ ... starting time of the r^{th} stage on machine M_i $(i = 1, \ldots, n; r = 0, 1, ...)$
- a_{ij} ... time M_j needs to prepare the component for M_i

- Machines $M_1, ..., M_n$ work interactively and in stages
- $x_i(r)$ . . . starting time of the r^{th} stage on machine M_i $(i = 1, \ldots, n; r = 0, 1, ...)$
- a_{ij} . . . time M_j needs to prepare the component for M_i
- $x_i(r+1) = max(x_1(r) + a_{i1}, \ldots, x_n(r) + a_{in})$ $(i = 1, \ldots, n; r = 0, 1, ...)$

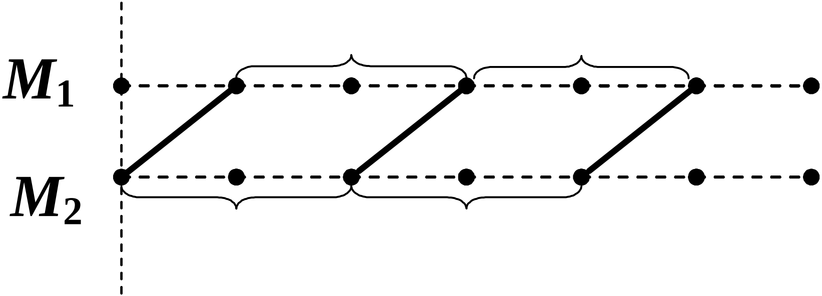
- Machines $M_1, ..., M_n$ work interactively and in stages
- $x_i(r)$ ... starting time of the r^{th} stage on machine M_i $(i = 1, \ldots, n; r = 0, 1, ...)$
- a_{ij} ... time M_j needs to prepare the component for M_i
- $x_i(r+1) = max(x_1(r) + a_{i1}, \ldots, x_n(r) + a_{in})$ $(i = 1, \ldots, n; r = 0, 1, ...)$
- $x_i(r+1) = \sum_k^{\oplus} a_{ik} \otimes x_k(r) \quad (i = 1, \ldots, n; r = 0, 1, ...)$

- Machines $M_1, ..., M_n$ work interactively and in stages
- $x_i(r)$ ... starting time of the r^{th} stage on machine M_i $(i = 1, \ldots, n; r = 0, 1, ...)$
- a_{ij} ... time M_j needs to prepare the component for M_i
- $x_i(r+1) = max(x_1(r) + a_{i1}, \ldots, x_n(r) + a_{in})$ $(i = 1, \ldots, n; r = 0, 1, ...)$
- $x_i(r+1) = \sum_k^{\oplus} a_{ik} \otimes x_k(r) \quad (i = 1, \ldots, n; r = 0, 1, ...)$
- $x(r+1) = A \otimes x(r) \quad (r = 0, 1, \ldots)$

- Machines $M_1, ..., M_n$ work interactively and in stages
- $x_i(r)$ ... starting time of the r^{th} stage on machine M_i $(i = 1, \ldots, n; r = 0, 1, ...)$
- a_{ij} ... time M_j needs to prepare the component for M_i
- $x_i(r+1) = max(x_1(r) + a_{i1}, \ldots, x_n(r) + a_{in})$ $(i = 1, \ldots, n; r = 0, 1, ...)$
- $x_i(r+1) = \sum_k^{\oplus} a_{ik} \otimes x_k(r) \quad (i = 1, \ldots, n; r = 0, 1, ...)$
- $x(r+1) = A \otimes x(r) \quad (r = 0, 1, \ldots)$
- $A : x(0) \to x(1) \to x(2) \to ...$

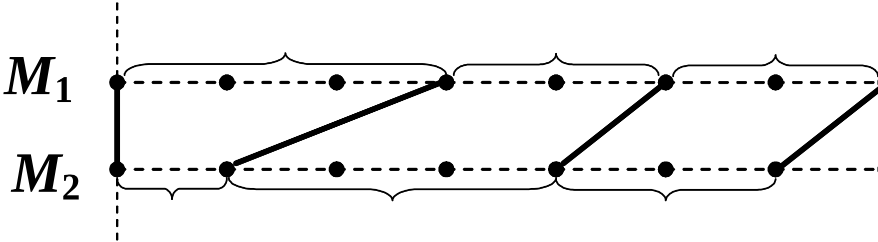
$$A : x(0) \longrightarrow x(1) \longrightarrow x(2) \longrightarrow \ldots$$

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 5 \\ 4 \end{pmatrix} \rightarrow \begin{pmatrix} 7 \\ 6 \end{pmatrix} \rightarrow \ldots$$



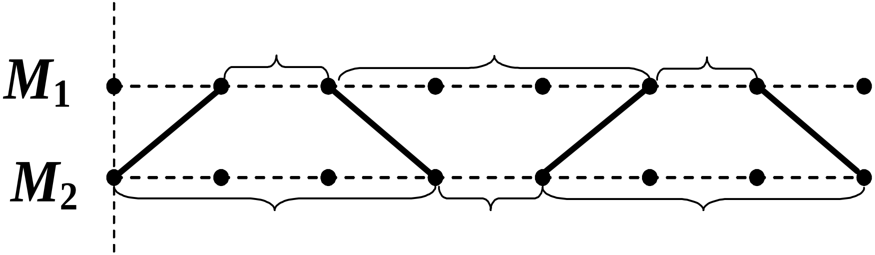
$\lambda = 2, r_0 = 0$

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 3 \\ 1 \end{pmatrix} \to \begin{pmatrix} 5 \\ 4 \end{pmatrix} \to \begin{pmatrix} 7 \\ 6 \end{pmatrix} \to \dots$$



$\lambda = 2, r_0 = 2$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 5 \\ 4 \end{pmatrix} \rightarrow \begin{pmatrix} 6 \\ 7 \end{pmatrix} \rightarrow \begin{pmatrix} 9 \\ 8 \end{pmatrix} \rightarrow \ldots$$



- Will the MPIS work in/reach a *steady regime* (that is, will it move forward in regular steps)?

- Will the MPIS work in/reach a *steady regime* (that is, will it move forward in regular steps)?
- Equivalently, is there a λ and an r_0 such that

$$x(r+1) = \lambda \otimes x(r) \quad (r \geq r_0)?$$

- Will the MPIS work in/reach a *steady regime* (that is, will it move forward in regular steps)?
- Equivalently, is there a λ and an r_0 such that

$$x(r+1) = \lambda \otimes x(r) \quad (r \geq r_0)?$$

-

$$x(r+1) = A \otimes x(r) \quad (r = 0, 1, \ldots)$$

- Will the MPIS work in/reach a *steady regime* (that is, will it move forward in regular steps)?
- Equivalently, is there a λ and an r_0 such that

$$x(r+1) = \lambda \otimes x(r) \quad (r \geq r_0)?$$

- $$x(r+1) = A \otimes x(r) \quad (r = 0, 1, \dots)$$

- MPIS reaches a steady regime if and only if for some λ and r, $x(r)$ is a solution to

$$A \otimes x = \lambda \otimes x$$

- Will the MPIS work in/reach a *steady regime* (that is, will it move forward in regular steps)?
- Equivalently, is there a λ and an r_0 such that

$$x(r+1) = \lambda \otimes x(r) \quad (r \geq r_0)?$$

- $$x(r+1) = A \otimes x(r) \quad (r = 0, 1, \ldots)$$

- MPIS reaches a steady regime if and only if for some λ and r, $x(r)$ is a solution to

$$A \otimes x = \lambda \otimes x$$

- Since

$$x(r) = A \otimes x(r-1) = A^2 \otimes x(r-2) = \ldots = A^r \otimes x(0),$$

a steady regime is reached if and only if $A^r \otimes x(0)$ "hits" an eigenvector of A for some r.

Given: $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$

- **Problem 1:** Find a $\lambda \in \mathbb{R}$ (*eigenvalue*) and an $x \in \overline{\mathbb{R}}^{n}$ (*eigenvector*) so that

$$A \otimes x = \lambda \otimes x.$$

Given: $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$

- **Problem 1:** Find a $\lambda \in \mathbb{R}$ (*eigenvalue*) and an $x \in \overline{\mathbb{R}}^n$ (*eigenvector*) so that

$$A \otimes x = \lambda \otimes x.$$

- **Problem 2**: Given an $x \in \overline{\mathbb{R}}^n$, is there a k such that $A^k \otimes x$ is an eigenvector of A?

Given: $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$

- **Problem 1:** Find a $\lambda \in \mathbb{R}$ (*eigenvalue*) and an $x \in \overline{\mathbb{R}}^n$ (*eigenvector*) so that

$$A \otimes x = \lambda \otimes x.$$

- **Problem 2**: Given an $x \in \overline{\mathbb{R}}^n$, is there a k such that $A^k \otimes x$ is an eigenvector of A?
- **Problem 3:** Is A *robust*? (That is for **every** $x \in \overline{\mathbb{R}}^n$, $x \neq \varepsilon$, there is a k such that $A^k \otimes x$ is an eigenvector of A)

- $A \otimes x = \lambda \otimes x$ *(steady regime immediately)* ... $O\left(n^3\right)$

- $A \otimes x = \lambda \otimes x$ (*steady regime immediately*) ... $O\left(n^3\right)$
- $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x, k \geq k_0$ (*reachability of a steady regime from a given start time vector*) ... Sergeev: $O\left(n^3 \log n\right)$

- $A \otimes x = \lambda \otimes x$ (*steady regime immediately*) ... $O\left(n^3\right)$
- $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x, k \geq k_0$ (*reachability of a steady regime from a given start time vector*) ... Sergeev: $O\left(n^3 \log n\right)$
- *Robustness* (*reachability of a steady regime from any start time vector*) ... polynomial algorithm

- $A \otimes x = \lambda \otimes x$ *(steady regime immediately)* ... $O\left(n^3\right)$
- $A^{k+1} \otimes x = \lambda \otimes A^k \otimes x, k \geq k_0$ *(reachability of a steady regime from a given start time vector)* ... Sergeev: $O\left(n^3 \log n\right)$
- *Robustness (reachability of a steady regime from any start time vector)* ... polynomial algorithm
- *Max-linear programming* ... pseudopolynomial algorithm

$$c^T \otimes x \longrightarrow \min\left(\text{or }\ \max\right)$$

s.t.

$$A \otimes x \oplus c = B \otimes x \oplus d$$

- $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}, N = \{1, ..., n\}$

- $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}, N = \{1, ..., n\}$
- $D_A = (N, \{(i,j); a_{ij} > \varepsilon\}, (a_{ij}))$ ... *digraph associated with A*

- $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}, N = \{1, ..., n\}$
- $D_A = (N, \{(i,j); a_{ij} > \varepsilon\}, (a_{ij}))$ ... *digraph associated with* A
- *Maximum cycle mean of* A :

$$\lambda(A) = \max\left\{\frac{a_{i_1i_2} + a_{i_2i_3} + ... + a_{i_ki_1}}{k}; i_1, ..., i_k \in N\right\}$$

- $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}, N = \{1, ..., n\}$
- $D_A = (N, \{(i,j); a_{ij} > \varepsilon\}, (a_{ij}))$ ... *digraph associated with* A
- *Maximum cycle mean of* A :

$$\lambda(A) = \max\left\{\frac{a_{i_1 i_2} + a_{i_2 i_3} + ... + a_{i_k i_1}}{k}; i_1, ..., i_k \in N\right\}$$

- $\Gamma(A) = A \oplus A^2 \oplus ... \oplus A^n$ (*metric matrix*)

- Given A, find all λ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$

- Given A, find all λ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$
- For any A, $\lambda(A)$ is

- Given A, find all λ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$
- For any A, $\lambda(A)$ is
  - an eigenvalue of A

- Given A, find all λ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$
- For any A, $\lambda(A)$ is
  - an eigenvalue of A
  - the greatest eigenvalue of A

- Given A, find all λ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$
- For any A, $\lambda(A)$ is
    - an eigenvalue of A
    - the greatest eigenvalue of A
    - the only eigenvalue of A whose corresponding eigenvectors may be finite

- Given A, find all λ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$
- For any A, $\lambda(A)$ is
  - an eigenvalue of A
  - the greatest eigenvalue of A
  - the only eigenvalue of A whose corresponding eigenvectors may be finite
  - the unique eigenvalue if A is irreducible

- Given A, find all λ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$
- For any A, $\lambda(A)$ is
  - an eigenvalue of A
  - the greatest eigenvalue of A
  - the only eigenvalue of A whose corresponding eigenvectors may be finite
  - the unique eigenvalue if A is irreducible
- Every eigenvalue of A is the maximum cycle mean for some principal submatrix

- Given A, find all λ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$
- For any A, $\lambda(A)$ is
    - an eigenvalue of A
    - the greatest eigenvalue of A
    - the only eigenvalue of A whose corresponding eigenvectors may be finite
    - the unique eigenvalue if A is irreducible
- Every eigenvalue of A is the maximum cycle mean for some principal submatrix
- If A is irreducible then the basis of the eigenspace can (easily) be found among the columns of $\Gamma\left(\left(\lambda\left(A\right)\right)^{-1} \otimes A\right)$ with zero diagonal entries

- If A is reducible - *Frobenius Normal Form* (FNF):

- If A is reducible - *Frobenius Normal Form* (FNF):
- $$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}$$, $A_{11}, \ldots, A_{rr}$ irreducible

- If A is reducible - *Frobenius Normal Form* (FNF):
- $$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}, A_{11}, ..., A_{rr} \text{ irreducible}$$
- $N_1, N_2, ..., N_r$ ... node sets of SCC of D_A

- If A is reducible - *Frobenius Normal Form* (FNF):
- $$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}, A_{11}, ..., A_{rr} \text{ irreducible}$$
- $N_1, N_2, ..., N_r$ ... node sets of SCC of D_A
- $N_i \longrightarrow N_j$ ... there is a path from N_i to N_j in D_A

- If A is reducible - *Frobenius Normal Form* (FNF):
- $$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}$$, $A_{11}, ..., A_{rr}$ irreducible
- $N_1, N_2, ..., N_r$ ... node sets of SCC of D_A
- $N_i \longrightarrow N_j$ ... there is a path from N_i to N_j in D_A
- $\Lambda(A) = \{\lambda(A_{ii}); \lambda(A_{ii}) \geq \lambda(A_{jj}) \text{ if } N_j \rightarrow N_i\}$ (Gaubert, Bapat)

- If A is reducible - *Frobenius Normal Form* (FNF):

- $$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}, A_{11}, ..., A_{rr} \text{ irreducible}$$

- $N_1, N_2, ..., N_r$ ... node sets of SCC of D_A
- $N_i \longrightarrow N_j$ ... there is a path from N_i to N_j in D_A
- $\Lambda(A) = \{\lambda(A_{ii}); \lambda(A_{ii}) \geq \lambda(A_{jj}) \text{ if } N_j \to N_i\}$ (Gaubert, Bapat)
- N_i is called *spectral* if $\lambda(A_{ii}) \geq \lambda(A_{jj})$ whenever $N_j \to N_i$

- $A = \left(\begin{array}{|cc|c|cc|c|}
\hline
0 & 3 & & & & \\
1 & 1 & & & & \\
\hline
 & & 4 & & & \\
\hline
 & & 0 & 0 & 1 & \\
 & & & -1 & 2 & \\
\hline
 & & & 1 & & 5 \\
\hline
\end{array}\right)$

- $A = \left(\begin{array}{|cc|c|cc|c|} \hline 0 & 3 & & & & \\ 1 & 1 & & & & \\ \hline & & 4 & & & \\ \hline & & 0 & 0 & 1 & \\ & & & -1 & 2 & \\ \hline & & & 1 & & 5 \\ \hline \end{array} \right)$

- $\lambda(A_{11}) = 2, \lambda(A_{22}) = 4, \lambda(A_{33}) = 2, \lambda(A_{44}) = 5, r = 4$

- $$A = \left(\begin{array}{cc|c|cc|c} 0 & 3 & & & & \\ 1 & 1 & & & & \\ \hline & & 4 & & & \\ \hline & & 0 & 0 & 1 & \\ & & & -1 & 2 & \\ \hline & & & 1 & & 5 \end{array}\right)$$

- $\lambda(A_{11}) = 2, \lambda(A_{22}) = 4, \lambda(A_{33}) = 2, \lambda(A_{44}) = 5, r = 4$
- $\Lambda(A) = \{2, 5\}$

- $A = \left(\begin{array}{cc|c|cc|c} 0 & 3 & & & & \\ 1 & 1 & & & & \\ \hline & & 4 & & & \\ \hline & & 0 & 0 & 1 & \\ & & & -1 & 2 & \\ \hline & & & 1 & & 5 \end{array}\right)$
- $\lambda(A_{11}) = 2, \lambda(A_{22}) = 4, \lambda(A_{33}) = 2, \lambda(A_{44}) = 5, r = 4$
- $\Lambda(A) = \{2, 5\}$
- $\lambda(A) = 5$

- $A = \left(\begin{array}{|cc|c|cc|c|} \hline 0 & 3 & & & & \\ 1 & 1 & & & & \\ \hline & & 4 & & & \\ \hline & & 0 & 0 & 1 & \\ & & & -1 & 2 & \\ \hline & & & 1 & & 5 \\ \hline \end{array}\right)$
- $\lambda(A_{11}) = 2, \lambda(A_{22}) = 4, \lambda(A_{33}) = 2, \lambda(A_{44}) = 5, r = 4$
- $\Lambda(A) = \{2, 5\}$
- $\lambda(A) = 5$
- N_1, N_4 are spectral; N_2, N_3 are not

- *Cyclicity* of a strongly connected digraph = g.c.d. of the lengths of its cycles

- *Cyclicity* of a strongly connected digraph = g.c.d. of the lengths of its cycles
- *Cyclicity* of a digraph = l.c.m. of cyclicities of its SSC

- *Cyclicity* of a strongly connected digraph = g.c.d. of the lengths of its cycles
- *Cyclicity* of a digraph = l.c.m. of cyclicities of its SSC
- Let $A \in \overline{\mathbb{R}}^{n \times n}$

- *Cyclicity* of a strongly connected digraph = g.c.d. of the lengths of its cycles
- *Cyclicity* of a digraph = l.c.m. of cyclicities of its SSC
- Let $A \in \overline{\mathbb{R}}^{n \times n}$
  - *Critical cycle* of A: any cycle whose mean is $\lambda(A)$

- *Cyclicity* of a strongly connected digraph = g.c.d. of the lengths of its cycles
- *Cyclicity* of a digraph = l.c.m. of cyclicities of its SSC
- Let $A \in \overline{\mathbb{R}}^{n\times n}$
  - *Critical cycle* of A: any cycle whose mean is $\lambda(A)$
  - *Critical digraph* of A, $C(A)$, consists of nodes and arcs on critical cycles of A

- *Cyclicity* of a strongly connected digraph = g.c.d. of the lengths of its cycles
- *Cyclicity* of a digraph = l.c.m. of cyclicities of its SSC
- Let $A \in \overline{\mathbb{R}}^{n \times n}$
  - *Critical cycle* of A: any cycle whose mean is $\lambda(A)$
  - *Critical digraph* of A, $C(A)$, consists of nodes and arcs on critical cycles of A
  - *Cyclicity of a matrix* $A, \sigma(A)$, is the cyclicity of $C(A)$

- *Cyclicity* of a strongly connected digraph = g.c.d. of the lengths of its cycles
- *Cyclicity* of a digraph = l.c.m. of cyclicities of its SSC
- Let $A \in \overline{\mathbb{R}}^{n \times n}$
  - *Critical cycle* of A: any cycle whose mean is $\lambda(A)$
  - *Critical digraph* of A, $C(A)$, consists of nodes and arcs on critical cycles of A
  - *Cyclicity of a matrix* $A, \sigma(A)$, is the cyclicity of $C(A)$
  - A is *primitive* if $\sigma(A) = 1$

- (*Cyclicity Theorem, Cohen et al*) Every irreducible matrix A is *ultimately periodic with period* $\sigma(A)$, that is $\sigma(A)$ is the least integer p for which there exists a positive integer k_0 such that:

$$A^{k+p} = (\lambda(A))^p \otimes A^k \text{ for all } k \geq k_0.$$

- (*Cyclicity Theorem, Cohen et al*) Every irreducible matrix A is *ultimately periodic with period* $\sigma(A)$, that is $\sigma(A)$ is the least integer p for which there exists a positive integer k_0 such that:

$$A^{k+p} = (\lambda(A))^p \otimes A^k \text{ for all } k \geq k_0.$$

- If $A \neq \varepsilon$ is irreducible and primitive:

- (*Cyclicity Theorem, Cohen et al*) Every irreducible matrix A is *ultimately periodic with period* $\sigma(A)$, that is $\sigma(A)$ is the least integer p for which there exists a positive integer k_0 such that:

$$A^{k+p} = (\lambda(A))^p \otimes A^k \text{ for all } k \geq k_0.$$

- If $A \neq \varepsilon$ is irreducible and primitive:
  - $A^{k+1} = (\lambda(A)) \otimes A^k$ for all $k \geq k_0$

- (*Cyclicity Theorem, Cohen et al*) Every irreducible matrix A is *ultimately periodic with period* $\sigma(A)$, that is $\sigma(A)$ is the least integer p for which there exists a positive integer k_0 such that:

$$A^{k+p} = (\lambda(A))^p \otimes A^k \text{ for all } k \geq k_0.$$

- If $A \neq \varepsilon$ is irreducible and primitive:
    - $A^{k+1} = (\lambda(A)) \otimes A^k$ for all $k \geq k_0$
    - $A^{k+1} \otimes x = (\lambda(A)) \otimes A^k \otimes x \neq \varepsilon$ for all $k \geq k_0$ and $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$

- (*Cyclicity Theorem, Cohen et al*) Every irreducible matrix A is *ultimately periodic with period* $\sigma(A)$, that is $\sigma(A)$ is the least integer p for which there exists a positive integer k_0 such that:

$$A^{k+p} = (\lambda(A))^p \otimes A^k \text{ for all } k \geq k_0.$$

- If $A \neq \varepsilon$ is irreducible and primitive:
    - $A^{k+1} = (\lambda(A)) \otimes A^k$ for all $k \geq k_0$
    - $A^{k+1} \otimes x = (\lambda(A)) \otimes A^k \otimes x \neq \varepsilon$ for all $k \geq k_0$ and $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$
- An irreducible matrix $A \neq \varepsilon$ is robust if and only if A is primitive

- $$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}$$ ... FNF

- $$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}$$ ... FNF
- $N_1, \ldots, N_r$ ... classes of A

- $\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}$ ... FNF
- $N_1, ..., N_r$ ... classes of A
- $A_{ii} = A\left[N_i\right], i = 1, ..., r$

- $$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix} \text{ ... FNF}$$
- $N_1, ..., N_r$ ... classes of A
- $A_{ii} = A\left[N_i\right], i = 1, ..., r$
- N_i is called *trivial* if A_{ii} is the 1×1 matrix (ε)

Robustness criterion:

Theorem (PB, Cuninghame-Green, Gaubert)

If $A \in \overline{\mathbb{R}}^{n \times n}$ has no ε column, then A is robust if and only if

- *Every non-trivial class is spectral and primitive*

Robustness criterion:

Theorem (PB, Cuninghame-Green, Gaubert)

If $A \in \overline{\mathbb{R}}^{n \times n}$ has no ε column, then A is robust if and only if

- *Every non-trivial class is spectral and primitive*
- $\lambda(A_{ii}) = \lambda(A_{jj})$ *if* N_i, N_j *are non-trivial,* $N_i \not\to N_j$ *and* $N_j \not\to N_i$

- $$A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$$

- $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$
- $r = 3, \Lambda(A) = \{0, 1, 2\}, N_j = \{j\}, j = 1, 2, 3$

- $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$
- $r = 3, \Lambda(A) = \{0, 1, 2\}, N_j = \{j\}, j = 1, 2, 3$
- $$A : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \longrightarrow$$

  will never reach an eigenvector

- $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$
- $r = 3, \Lambda(A) = \{0, 1, 2\}, N_j = \{j\}, j = 1, 2, 3$
- $$A : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \longrightarrow$$

  will never reach an eigenvector
- Note: $N_1 \nrightarrow N_2$ and $N_2 \nrightarrow N_1$ but $\lambda(N_1) \neq \lambda(N_2)$

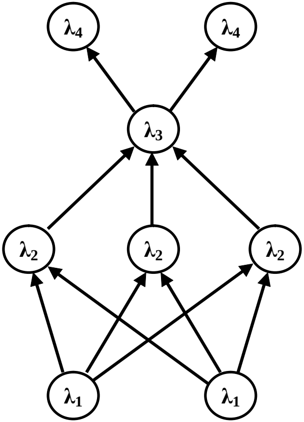
- $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$

- $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$
- $r = 3, \Lambda(A) = \{0, 2\}, N_j = \{j\}, j = 1, 2, 3$

- $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$
- $r = 3, \Lambda(A) = \{0, 2\}, N_j = \{j\}, j = 1, 2, 3$
- A is robust

- $A = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$
- $r = 3, \Lambda(A) = \{0, 2\}, N_j = \{j\}, j = 1, 2, 3$
- A is robust
- $$A: \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ \varepsilon \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 \\ \varepsilon \\ 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 6 \\ \varepsilon \\ 4 \end{pmatrix} \longrightarrow \ldots$$

Example of the condensation digraph of a robust matrix with $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$:



- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$

- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$
- $\Lambda(A, B)$ ... set of *generalised eigenvalues*

- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$

- $\Lambda(A, B)$ ... set of *generalised eigenvalues*
- (*Sergeev*) Any union of closed real intervals is $\Lambda(A, B)$ for some A and B

- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$

- $\Lambda(A, B)$ ... set of *generalised eigenvalues*
- (*Sergeev*) Any union of closed real intervals is $\Lambda(A, B)$ for some A and B
- $V(A, B)$ ... set of *generalised eigenvectors*

- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$

- $\Lambda(A, B)$ ... set of *generalised eigenvalues*
- (*Sergeev*) Any union of closed real intervals is $\Lambda(A, B)$ for some A and B
- $V(A, B)$ ... set of *generalised eigenvectors*
- Special cases (A and B finite):

- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$
- $\Lambda(A, B)$ ... set of *generalised eigenvalues*
- (*Sergeev*) Any union of closed real intervals is $\Lambda(A, B)$ for some A and B
- $V(A, B)$ ... set of *generalised eigenvectors*
- Special cases (A and B finite):
  - (*Binding & Volkmer*) (A, B) and (A^T, B^T) solvable $\Longrightarrow$ $\Lambda(A, B) = \{\lambda\} = \Lambda(A^T, B^T)$ for some λ

- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$

- $\Lambda(A, B)$ ... set of *generalised eigenvalues*
- (*Sergeev*) Any union of closed real intervals is $\Lambda(A, B)$ for some A and B
- $V(A, B)$ ... set of *generalised eigenvectors*
- Special cases (A and B finite):
  - (*Binding & Volkmer*) (A, B) and (A^T, B^T) solvable $\Longrightarrow$ $\Lambda(A, B) = \{\lambda\} = \Lambda(A^T, B^T)$ for some λ
  - A and B symmetric $\Longrightarrow |\Lambda(A, B)| \leq 1$

- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$

- $\Lambda(A, B)$ ... set of *generalised eigenvalues*
- (*Sergeev*) Any union of closed real intervals is $\Lambda(A, B)$ for some A and B
- $V(A, B)$ ... set of *generalised eigenvectors*
- Special cases (A and B finite):
  - (*Binding & Volkmer*) (A, B) and (A^T, B^T) solvable $\Longrightarrow$ $\Lambda(A, B) = \{\lambda\} = \Lambda(A^T, B^T)$ for some λ
  - A and B symmetric $\Longrightarrow |\Lambda(A, B)| \leq 1$
  - Every common eigenvector of A and B (if any) is a generalised eigenvector for A and B

- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$

- $\Lambda(A, B)$ ... set of *generalised eigenvalues*
- (*Sergeev*) Any union of closed real intervals is $\Lambda(A, B)$ for some A and B
- $V(A, B)$ ... set of *generalised eigenvectors*
- Special cases (A and B finite):
  - (*Binding & Volkmer*) (A, B) and (A^T, B^T) solvable $\Longrightarrow$ $\Lambda(A, B) = \{\lambda\} = \Lambda(A^T, B^T)$ for some λ
  - A and B symmetric $\Longrightarrow |\Lambda(A, B)| \leq 1$
  - Every common eigenvector of A and B (if any) is a generalised eigenvector for A and B
  - (*Schneider*) A and B commute $\Longrightarrow$ A and B have a common eigenvector

- $A \otimes x = \lambda \otimes B \otimes x, \quad A, B \in \overline{\mathbb{R}}^{m \times n}$

- $\Lambda(A, B)$ ... set of *generalised eigenvalues*
- (*Sergeev*) Any union of closed real intervals is $\Lambda(A, B)$ for some A and B
- $V(A, B)$ ... set of *generalised eigenvectors*
- Special cases (A and B finite):
  - (*Binding & Volkmer*) (A, B) and (A^T, B^T) solvable $\Longrightarrow$ $\Lambda(A, B) = \{\lambda\} = \Lambda(A^T, B^T)$ for some λ
  - A and B symmetric $\Longrightarrow |\Lambda(A, B)| \leq 1$
  - Every common eigenvector of A and B (if any) is a generalised eigenvector for A and B
  - (*Schneider*) A and B commute $\Longrightarrow$ A and B have a common eigenvector
  - A and B commute $\Longrightarrow \Lambda(A, B) = \{\lambda\} = \Lambda(A^T, B^T)$ for some λ

- $A, B \in \mathbb{R}^{n \times n}$

- $A, B \in \mathbb{R}^{n \times n}$
- $C = (c_{ij}) = (a_{ij} \otimes b_{ij}^{-1})$

- $A, B \in \mathbb{R}^{n \times n}$
- $C = (c_{ij}) = (a_{ij} \otimes b_{ij}^{-1})$
- $\Lambda(A, B) \subseteq [\max_i \min_j c_{ij}, \min_i \max_j c_{ij}]$

- $A, B \in \mathbb{R}^{n \times n}$
- $C = (c_{ij}) = (a_{ij} \otimes b_{ij}^{-1})$
- $\Lambda(A, B) \subseteq [\max_i \min_j c_{ij}, \min_i \max_j c_{ij}]$
- $L = \{\lambda; (\exists i, j)\, a_{ij} = \lambda \otimes b_{ij}\}, |L| \leq n^2$

- $A, B \in \mathbb{R}^{n \times n}$
- $C = (c_{ij}) = (a_{ij} \otimes b_{ij}^{-1})$
- $\Lambda(A, B) \subseteq [\max_i \min_j c_{ij}, \min_i \max_j c_{ij}]$
- $L = \{\lambda; (\exists i, j)\, a_{ij} = \lambda \otimes b_{ij}\}, |L| \leq n^2$
- $I = [\ell, u]$ is called *regular* if $L \cap I = \{\ell, u\}$

- $A, B \in \mathbb{R}^{n \times n}$
- $C = (c_{ij}) = (a_{ij} \otimes b_{ij}^{-1})$
- $\Lambda(A, B) \subseteq [\max_i \min_j c_{ij}, \min_i \max_j c_{ij}]$
- $L = \{\lambda; (\exists i, j)\, a_{ij} = \lambda \otimes b_{ij}\}, |L| \leq n^2$
- $I = [\ell, u]$ is called *regular* if $L \cap I = \{\ell, u\}$
- Wlog: $\lambda \in I$, that is $(\forall i, j \in N)\, a_{ij} \neq \lambda \otimes b_{ij}$

- $\mathbb{S} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}$

$$
\begin{aligned}
(a, a') \oplus (b, b') &= (a \oplus b, a' \oplus b') \\
(a, a') \otimes (b, b') &= (a \otimes b \oplus a' \otimes b', a \otimes b' \oplus a' \otimes b) \\
\ominus (a, b) &= (b, a) \\
|(a, b)| &= a \oplus b \\
x \ominus y &= x \oplus (\ominus y) \text{ for } x, y \in \mathbb{S}
\end{aligned}
$$

- $\mathbb{S} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}$

$$\begin{aligned}
(a, a') \oplus (b, b') &= (a \oplus b, a' \oplus b') \\
(a, a') \otimes (b, b') &= (a \otimes b \oplus a' \otimes b', a \otimes b' \oplus a' \otimes b) \\
\ominus (a, b) &= (b, a) \\
|(a, b)| &= a \oplus b \\
x \ominus y &= x \oplus (\ominus y) \text{ for } x, y \in \mathbb{S}
\end{aligned}$$

- Relation of *balance*: $(a, a') \bigtriangledown (b, b')$ iff $a \oplus b' = a' \oplus b$

- $\mathbb{S} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}$

$$
\begin{aligned}
(a, a') \oplus (b, b') &= (a \oplus b, a' \oplus b') \\
(a, a') \otimes (b, b') &= (a \otimes b \oplus a' \otimes b', a \otimes b' \oplus a' \otimes b) \\
\ominus (a, b) &= (b, a) \\
|(a, b)| &= a \oplus b \\
x \ominus y &= x \oplus (\ominus y) \text{ for } x, y \in \mathbb{S}
\end{aligned}
$$

- Relation of *balance*: $(a, a') \bigtriangledown (b, b')$ iff $a \oplus b' = a' \oplus b$
- (a, b) is *sign-positive* [*sign-negative*] iff $a > b$ $[a < b]$ or $a = b = \varepsilon$

- $\mathbb{S} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}$

$$\begin{aligned}
(a, a') \oplus (b, b') &= (a \oplus b, a' \oplus b') \\
(a, a') \otimes (b, b') &= (a \otimes b \oplus a' \otimes b', a \otimes b' \oplus a' \otimes b) \\
\ominus (a, b) &= (b, a) \\
|(a, b)| &= a \oplus b \\
x \ominus y &= x \oplus (\ominus y) \text{ for } x, y \in \mathbb{S}
\end{aligned}$$

- Relation of *balance*: $(a, a') \bigtriangledown (b, b')$ iff $a \oplus b' = a' \oplus b$
- (a, b) is *sign-positive* [*sign-negative*] iff $a > b$ $[a < b]$ or $a = b = \varepsilon$
- (a, b) is *balanced* iff $a = b$, otherwise it is called *unbalanced*

- For $A = (a_{ij}) \in \mathbb{R}^{n \times n}$

$$maper(A) \overset{df}{=} \sum_{\pi}^{\oplus} \prod_{i}^{\otimes} a_{i,\pi(i)} = \max_{\pi} \sum_{i} a_{i,\pi(i)}$$

- For $A = (a_{ij}) \in \mathbb{R}^{n \times n}$

$$maper(A) \overset{df}{=} \sum_{\pi}^{\oplus} \prod_{i}^{\otimes} a_{i,\pi(i)} = \max_{\pi} \sum_{i} a_{i,\pi(i)}$$

-

$$ap(A) = \left\{ \pi \in P_n;\, maper(A) = \sum_{i} a_{i,\pi(i)} \right\}$$

- For $A = (a_{ij}) \in \mathbb{R}^{n \times n}$

$$maper(A) \stackrel{df}{=} \sum_{\pi}^{\oplus} \prod_{i}^{\otimes} a_{i,\pi(i)} = \max_{\pi} \sum_{i} a_{i,\pi(i)}$$

- $$ap(A) = \left\{ \pi \in P_n ; maper(A) = \sum_{i} a_{i,\pi(i)} \right\}$$

- For $C = (c_{ij}) \in \mathbb{S}^{n \times n}$

$$det(C) = \sum\nolimits_{\pi}^{\oplus} \left(sgn(\pi) \otimes \prod\nolimits_{i}^{\otimes} c_{i,\pi(i)} \right) = \left(d^{+}(C), d^{-}(C) \right)$$

- For $A = (a_{ij}) \in \mathbb{R}^{n \times n}$

$$maper(A) \overset{df}{=} \sum_{\pi}^{\oplus} \prod_{i}^{\otimes} a_{i,\pi(i)} = \max_{\pi} \sum_{i} a_{i,\pi(i)}$$

- $$ap(A) = \left\{ \pi \in P_n ; maper(A) = \sum_{i} a_{i,\pi(i)} \right\}$$

- For $C = (c_{ij}) \in \mathbb{S}^{n \times n}$

$$det(C) = \sum\nolimits_{\pi}^{\oplus} \left(sgn(\pi) \otimes \prod\nolimits_{i}^{\otimes} c_{i,\pi(i)} \right) = \left(d^{+}(C), d^{-}(C) \right)$$

- $$|\det(C)| = d^{+}(C) \oplus d^{-}(C) = maper(|C|)$$

- $\left(\exists x \in \overline{\mathbb{R}}^n, x \neq \varepsilon\right) A \otimes x = B \otimes x \Longleftrightarrow$
  $(\exists y \in \mathbb{S}^n, y \neq \varepsilon,\ y \text{ sign positive})\ (A \ominus B) \otimes y \bigtriangledown \varepsilon$

- $\left(\exists x \in \overline{\mathbb{R}}^n, x \neq \varepsilon\right) A \otimes x = B \otimes x \Longleftrightarrow$
$(\exists y \in \mathbb{S}^n, y \neq \varepsilon,\ y \text{ sign positive})\ (A \ominus B) \otimes y \,\nabla\, \varepsilon$
- (M.Plus) Let $C \in \mathbb{S}^{n \times n}$. Then the system of balances $C \otimes y \,\nabla\, \varepsilon$ has a signed non-trivial solution if and only if $det(C) \,\nabla\, \varepsilon$.

- $\left(\exists x \in \overline{\mathbb{R}}^n, x \neq \varepsilon\right) A \otimes x = B \otimes x \Longleftrightarrow$
  $(\exists y \in \mathbb{S}^n, y \neq \varepsilon,\ y$ sign positive$)$ $(A \ominus B) \otimes y \mathrel{\nabla} \varepsilon$
- (M.Plus) Let $C \in \mathbb{S}^{n\times n}$. Then the system of balances $C \otimes y \mathrel{\nabla} \varepsilon$ has a signed non-trivial solution if and only if $det(C) \mathrel{\nabla} \varepsilon$.
- (PB, Gaubert) Let $A, B \in \overline{\mathbb{R}}^{n\times n}$ and $C = A \ominus B$. Then a necessary condition that the system $A \otimes x = B \otimes x$ have a non-trivial solution is that C has balanced determinant.

- $\left(\exists x \in \overline{\mathbb{R}}^n, x \neq \varepsilon\right) A \otimes x = B \otimes x \iff$ $(\exists y \in \mathbb{S}^n, y \neq \varepsilon,\ y \text{ sign positive})\ (A \ominus B) \otimes y \ \nabla\ \varepsilon$
- (M.Plus) Let $C \in \mathbb{S}^{n\times n}$. Then the system of balances $C \otimes y \ \nabla\ \varepsilon$ has a signed non-trivial solution if and only if $det(C) \ \nabla\ \varepsilon$.
- (PB, Gaubert) Let $A, B \in \overline{\mathbb{R}}^{n\times n}$ and $C = A \ominus B$. Then a necessary condition that the system $A \otimes x = B \otimes x$ have a non-trivial solution is that C has balanced determinant.
- Let $A, B \in \overline{\mathbb{R}}^{n\times n}$ and $C(\lambda) = A \ominus \lambda \otimes B$. Then a necessary condition that the system $A \otimes x = \lambda \otimes B \otimes x$ have a non-trivial solution is that $C(\lambda)$ has balanced determinant, that is if $\det(C(\lambda)) = (d^+(\lambda), d^-(\lambda))$:

$$\lambda \in \Lambda(A, B) \Longrightarrow d^+(\lambda) = d^-(\lambda)$$

- $$d^{+}(\lambda) \oplus d^{-}(\lambda) = |\det(C(\lambda))| = maper(|C(\lambda)|)$$

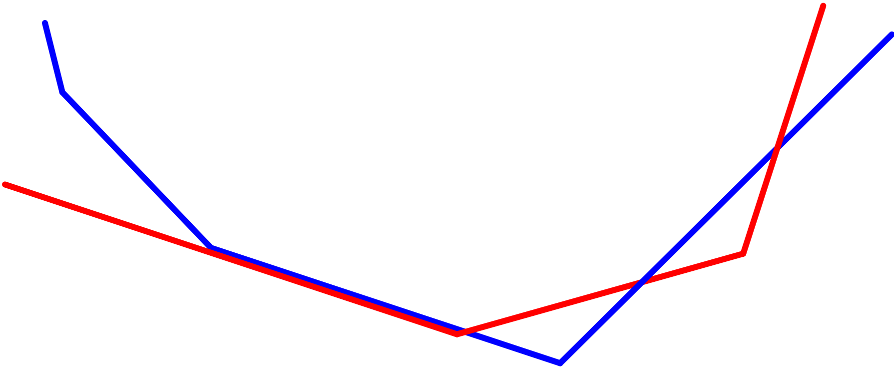
- $$d^{+}(\lambda) \oplus d^{-}(\lambda) = |\det(C(\lambda))| = maper(|C(\lambda)|)$$
- $d^{+}(\lambda)$ and $d^{-}(\lambda)$ are maxpolynomials in λ (hence piecewise linear and convex functions) containing at most $n+1$ powers of λ between 0 and n.

- $$d^{+}(\lambda) \oplus d^{-}(\lambda) = |\det(C(\lambda))| = maper(|C(\lambda)|)$$
- $d^{+}(\lambda)$ and $d^{-}(\lambda)$ are maxpolynomials in λ (hence piecewise linear and convex functions) containing at most $n+1$ powers of λ between 0 and n.
- It is not known how to find $d^{+}(\lambda)$ and $d^{-}(\lambda)$ individually but
  $$|C(\lambda)| = (a_{ij} \oplus \lambda \otimes b_{ij}) = (c_{ij}(\lambda))$$
  and $maper(|C(\lambda)|) = d^{+}(\lambda) \oplus d^{-}(\lambda)$ can be found in $< O(n^4)$ time.

- $$d^{+}(\lambda) \oplus d^{-}(\lambda) = |\det(C(\lambda))| = maper(|C(\lambda)|)$$
- $d^{+}(\lambda)$ and $d^{-}(\lambda)$ are maxpolynomials in λ (hence piecewise linear and convex functions) containing at most $n+1$ powers of λ between 0 and n.
- It is not known how to find $d^{+}(\lambda)$ and $d^{-}(\lambda)$ individually but
  $$|C(\lambda)| = (a_{ij} \oplus \lambda \otimes b_{ij}) = (c_{ij}(\lambda))$$
  and $maper(|C(\lambda)|) = d^{+}(\lambda) \oplus d^{-}(\lambda)$ can be found in $< O(n^4)$ time.
- For finding ALL values of $\lambda \in I$ satisfying $d^{+}(\lambda) = d^{-}(\lambda)$ we now only need to check this for the corners of $d^{+}(\lambda) \oplus d^{-}(\lambda)$ and one value between two consecutive corners

4.2 Generalised eigenproblem (GEP)

Narrowing the search for generalised eigenvalues



- $C = (c_{ij}) \in \mathbb{S}^{n \times n} \longrightarrow \tilde{C} = (\tilde{c}_{ij})$ is $(0, 1, -1)$ satisfying

$$\begin{aligned}
\tilde{c}_{ij} &= 1 \ \text{ if } j = \pi(i) \ \text{ for } \pi \in ap(|C|) \text{ and } c_{ij} \text{ is sign-positive,} \\
\tilde{c}_{ij} &= -1 \ \text{ if } j = \pi(i) \ \text{ for } \pi \in ap(|C|) \text{ and } c_{ij} \text{ is sign-negative,} \\
\tilde{c}_{ij} &= 0 \ \text{ else.}
\end{aligned}$$

- $C = (c_{ij}) \in \mathbb{S}^{n\times n} \longrightarrow \tilde{C} = (\tilde{c}_{ij})$ is $(0, 1, -1)$ satisfying

$$\begin{aligned}
\tilde{c}_{ij} &= 1 \text{ if } j = \pi(i) \text{ for } \pi \in ap(|C|) \text{ and } c_{ij} \text{ is sign-positive,} \\
\tilde{c}_{ij} &= -1 \text{ if } j = \pi(i) \text{ for } \pi \in ap(|C|) \text{ and } c_{ij} \text{ is sign-negative,} \\
\tilde{c}_{ij} &= 0 \text{ else.}
\end{aligned}$$

- Let $C \in \mathbb{S}^{n\times n}$. A necessary condition that C have unbalanced determinant is that $\tilde{C}$ is SNS. If C has no balanced entry then this condition is also sufficient.

- $C = (c_{ij}) \in \mathbb{S}^{n\times n} \longrightarrow \tilde{C} = (\tilde{c}_{ij})$ is $(0, 1, -1)$ satisfying

$$\begin{aligned}
\tilde{c}_{ij} &= 1 \text{ if } j = \pi(i) \text{ for } \pi \in ap(|C|) \text{ and } c_{ij} \text{ is sign-positive,} \\
\tilde{c}_{ij} &= -1 \text{ if } j = \pi(i) \text{ for } \pi \in ap(|C|) \text{ and } c_{ij} \text{ is sign-negative,} \\
\tilde{c}_{ij} &= 0 \text{ else.}
\end{aligned}$$

- Let $C \in \mathbb{S}^{n\times n}$. A necessary condition that C have unbalanced determinant is that $\tilde{C}$ is SNS. If C has no balanced entry then this condition is also sufficient.
- If $\lambda \in I$ then $A \ominus \lambda \otimes B$ has no balanced entry.

- $C = (c_{ij}) \in \mathbb{S}^{n \times n} \longrightarrow \tilde{C} = (\tilde{c}_{ij})$ is $(0, 1, -1)$ satisfying

$$\begin{aligned} \tilde{c}_{ij} &= 1 \text{ if } j = \pi(i) \text{ for } \pi \in ap(|C|) \text{ and } c_{ij} \text{ is sign-positive,} \\ \tilde{c}_{ij} &= -1 \text{ if } j = \pi(i) \text{ for } \pi \in ap(|C|) \text{ and } c_{ij} \text{ is sign-negative,} \\ \tilde{c}_{ij} &= 0 \text{ else.} \end{aligned}$$

- Let $C \in \mathbb{S}^{n \times n}$. A necessary condition that C have unbalanced determinant is that $\tilde{C}$ is SNS. If C has no balanced entry then this condition is also sufficient.
- If $\lambda \in I$ then $A \ominus \lambda \otimes B$ has no balanced entry.
- Let $A, B \in \mathbb{R}^{n \times n}$ and $C(\lambda) = A \ominus \lambda \otimes B, \lambda \in I$. Then $C(\lambda)$ is balanced if and only if $\tilde{C}(\lambda)$ is not SNS.

CONCLUSION:
The set of all λ satisfying $d^{+}(\lambda) = d^{-}(\lambda)$ can be found in polynomial time.

THANK YOU

R.A.Cuninghame-Green 1960
N.N.Vorobyov 1967, Litvinov, Maslov, Kolokoltsov, Sobolevski...
M.Gondran, M.Minoux 1975
K.Zimmermann 1972
P.Butkovič 1977
R.E.Burkard, U.Zimmermann 1981
H.Schneider
G.Cohen, D.Dubois, J.-P.Quadrat, M.Viot 1983
K.Cechlárová, P.Szabó, J.Plávka 1985
G.-J.Olsder, C.Roos 1988, B.Heidergott 2000
R.D.Nussbaum 1991
M.Akian 1998, S.Gaubert 1992, C.Walsh 2001
R.B.Bapat, D.Stanford, P.van den Driessche 1993
M.Gavalec 1995
Tropical algebra from 1995: B.Sturmfels, M.Develin et al