Perron-Frobenius Theory and Positivity in Linear Algebra

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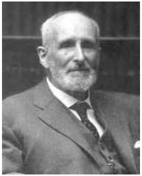
ALA - Novi Sad 2010 - In honour of Hans Schneider

May 26, 2010



Oskar Perron (1880-1975)

Georg Frobenius (1849-1917)

Theorem (Perron-Frobenius)

Let A be a square matrix with nonnegative entries. Then the largest in modulus eigenvalue of A is nonnegative and has a nonnegative eigenvector. That is,

$$Ax = \rho(A)\, x,$$

where $x \neq 0$ is an entrywise nonnegative vector and

$$\rho(A) = \max\{|\lambda| \;:\; \lambda \in \sigma(A)\}.$$

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Theorem (Perron-Frobenius, irreducible matrix)

Let A be an *irreducible* square matrix with nonnegative entries. Then the largest in modulus eigenvalue of A is simple, positive and has a positive eigenvector.

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- The linear algebraic foundations of Markov's theory for stochastic matrices are first established by **Von Mises** and **Romanovsky** by means of Frobenius' 1912 paper.
- Long pause.... about 40 years long (with the exception of some work by Ostrowski and Wielandt) until a new graduate student in Edinburgh in 1950....

Hans Schneider - age $1 \pm \epsilon$

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Hans' Recollections

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- But Aitken reintroduced matrices and proved (perhaps following Romanovsky) that a stochastic matrix is regular iff 1 is simple and the lone eigenvalue on the unit circle.
- Combinatorics (irreducibility) is entirely absent from Aitken's notes.

Shaping Hans' Mathematical Life

Fragment from Aitken's notes:

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- Raising a matrix P to successive powers is laborious, and the discovery of n_0 may be long deferred. <u>Better criteria are to be found in the structure of P itself, its latent roots, vectors and canonical form...</u>

Hans' Current View of his Thesis

Hans embarks on the discovery eluded to by Aitken:

- Thesis partly in: The Elementary Divisors Associated with 0 of a Singular M-matrix, *Proc. Edinburgh Math. Soc., 10(1956), 108–122*.

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- He essentially introduces the notion of the reduced graph in terms of coefficients that can be 0 or 1.
- Applied it to concepts connected to the Jordan form he had heard Aitken discuss.

A (familiar) Theorem by Hans

THEOREM 2a

Let $A = [A_{ij}]$, $i, j = 1, \ldots, k$, be a singular M-matrix in standard form. Let S be the set of indices of singular A_{ii}. If $\alpha \in S$ and $R_{\alpha\beta} = 0$ whenever $\beta \in S$ and $\beta \neq \alpha$, then there exists a positive characteristic row vector

$$u' = (u'_1, \ldots, u'_k)$$

associated with 0, satisfying

$$\begin{aligned} u_i &= 0 \quad \text{when } R_{\alpha i} = 0 \\ u_i &> 0 \quad \text{when } R_{\alpha i} = 1, \end{aligned}$$

for $i = 1, \ldots, k$.

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- What other matrices satisfy the Perron-Frobenius theorem?
- Is there a wider mathematical context?
- Is there a converse theorem? More than "one"?
- What are some challenging questions?

Cone Nonnegativity

Replace $\mathbb{R}^n_+$ by a an appropriate convex set.

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Definition A convex set $K \subset \mathbb{R}^n$ is called a **proper cone** if

- $aK \subseteq K$ for all $a \geq 0$ (cone)
- $K \cap (-K) = \{0\}$ (pointed)
- $\operatorname{int} K \neq \emptyset$ (solid)
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Theorem (Perron-Frobenius)
Let K be a proper cone and suppose $AK \subset K$. Then $\rho(A)$ is an eigenvalue of A corresponding to an eigenvector $x \in K$.

Perron-Frobenius in Operator Algebras

Theorem (Perron-Frobenius)
Suppose ϕ is a positive linear map on a von Neumann algebra A. Let r denote the spectral radius of ϕ and A^+ the cone of positive elements of A. Then there exists nonzero $z \in A^+$ such that $\phi(z) = r\,z$.

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• A von Neumann algebra is a *-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator.

• Here, positive elements of A are ones of the form a^*a.

Perron-Frobenius in Functional Analysis

Function $f : \operatorname{int} \mathbb{R}^n_+ \longrightarrow \operatorname{int} \mathbb{R}^n_+$ is:

- **homogeneous** if $\forall \lambda > 0$ and $\forall x \in \operatorname{int} \mathbb{R}^n_+$, $f(\lambda x) = \lambda f(x)$;
- **monotone** if $\forall x \leq y \in \operatorname{int} \mathbb{R}^n_+$, $f(x) \leq f(y)$.

Given $u > 0$, $J \subseteq \{1, \ldots, n\}$, define

$$(u_J)_i = \begin{cases} u & \text{if } i \in J \\ 1 & \text{if } i \notin J \end{cases}$$

.

- **Directed graph** $G(f)$ on vertices $\{1, \ldots, n\}$ with edge $i \to j$ if

$$\lim_{u \to \infty} f_i(u_{\{j\}}) = \infty.$$

Theorem (Perron-Frobenius)
Let $f : \operatorname{int} \mathbb{R}^n_+ \longrightarrow \operatorname{int} \mathbb{R}^n_+$ be a homogeneous, monotone function. If $G(f)$ is strongly connected, then f has an eigenvector in $\operatorname{int} \mathbb{R}^n_+$.

Example: In contrast to linear irreducible case, eigenvector need not be "unique":

$$f(x_1, x_2) = \left(\max\{x_1, \frac{x_2}{2}\}, \max\{\frac{x_1}{2}, x_2\}\right)$$

has eigenvalue 1 with corresponding eigenspace

$$\{x \in \operatorname{int} \mathbb{R}^2_+ \mid x_1/2 \leq x_2 \leq 2x_1\}.$$

Perron-Frobenius in Max Algebra

Max Algebra: For nonnegative x, y and $A, B \in \mathbb{R}^{n\times n}$,

$$\begin{cases} x+y & \longrightarrow & x \oplus y = \max(x,y) \\ x\,y & \longrightarrow & x\,y \\ A\,B & \longrightarrow & A \otimes B \end{cases}$$

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Theorem (Max Perron-Frobenius) [Cunningham-Green 1962/1979, Gondrian-Minoux, 1977]

If A is an $n \times n$ nonnegative, irreducible matrix, then there exists $r > 0$, and a positive vector x such that for each $i = 1, \ldots, n$,

$$\max_j a_{ij} x_j = r x_i, \quad \text{(i.e., } A \otimes x = r\,x)$$

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Peter Butkovic: Max Alg. Primer (website) & upcoming book!

Example: Again, in contrast to linear irreducible case, eigenvector need not be "unique", e.g.,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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Perron-Frobenius in Matrix Polynomials

Theorem (Perron-Frobenius) [Psarrakos, T, LAA 2004] Let

$$L(\lambda) = I\lambda^m - A_{m-1}\lambda^{m-1} - \cdots - A_1\lambda - A_0,$$

where $A_j \geq 0$. Then the following hold:

(a) $\rho(L)$ is an eigenvalue of $L(\lambda)$;

(b) $L(\lambda)$ has an entrywise nonnegative eigenvector corresponding to $\rho(L)$;

(c) $\rho(L)$ is a nondecreasing function of the entries of A_j.

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Theorem (Krein-Ruttman)
Suppose $\rho(A)$ is an eigenvalue of A, and $\deg \rho(A) \geq \deg \lambda$ for every eigenvalue λ with $|\lambda| = \rho(A)$. Then there exists a proper cone K such that $AK \subset K$.

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- Degree of λ = size of largest Jordan block.
- Finite dimensional case of a theorem on Banach spaces.
- Constructive proof due to Birkhoff.

In search of an alternative converse

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Perron's Corollary "The properties of positive matrices hold for all nonnegative matrices A such that A^k has positive entries for some power k."

Insightful: Anticipated notions of irreducibility, primitivity etc.

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Insightful: Suggests notion of *Eventual Nonnegativity*

(S. Friedland, B. Zaslavsky, D. Handelman, B.-S. Tam, C.R. Johnson, P. Tarazaga, J. McDonald, D. Noutsos, A. Elhashash, D. Szyld et al.)

Some definitions

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Recall that $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$

Perron-Frobenius property

We say that $A \in \mathbb{R}^{n \times n}$ has

- the **Perron-Frobenius property** if $\rho(A) > 0$, $\rho(A) \in \sigma(A)$ and $\exists$ a nonnegative eigenvector corresponding to $\rho(A)$;
- the **strong Perron-Frobenius property** if, in addition to having the Perron-Frobenius property, $\rho(A)$ is simple,

$$\rho(A) > |\lambda| \quad \text{for all} \quad \lambda \in \sigma(A), \;\; \lambda \neq \rho(A)$$

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Note: Every $A \geq 0$ has the Perron-Frobenius property and every irreducible $A \geq 0$ has the strong Perron-Frobenius property.

Eventually positive matrices

Theorem For a matrix $A \in \mathbb{R}^{n\times n}$ the following are equivalent:

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First known explicit reference: [**Handelman**, J. Operator Theory, 1981].
Also [Johnson and Tarazaga, Positivity, 2004], and [Noutsos, LAA, 2006].

Eventually exponentially positive matrices

Theorem [Noutsos, T, SIMAX 2008] For a matrix $A \in \mathbb{R}^{n \times n}$ the following properties are equivalent:

(i) $\exists\, a \geq 0$ such that $A + aI$ and $A^T + aI$ have the strong Perron-Frobenius property.

(ii) $A + aI$ is eventually positive for some $a \geq 0$.

(iii) $A^T + aI$ is eventually positive for some $a \geq 0$.

(iv) A is eventually exponentially positive.

(v) A^T is eventually exponentially positive.

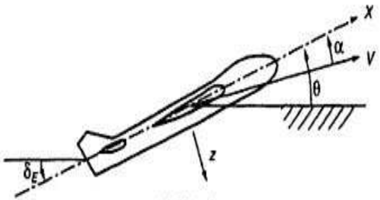
Application

- Control system $\dot{x}(t) = Ax(t) + Bu(t)$, A, B fixed, control u to be chosen.
- Feedback control $u(t) = Fx(t)$.
- System becomes $\dot{x}(t) = (A + BF)x(t)$.
- Solution is $x(t) = e^{t(A+BF)}x_0$.
- If F is chosen so that $A + BF$ is eventually positive, then $x(t)$ <u>becomes and remains nonnegative</u>.

F-16 longitudinal motion - Landing

Some of the controls available to a pilot vary; others are trimmed. The aerodynamic variables describing its motion:

$$\Delta u \text{ (Change in speed)}, \quad \alpha \text{ (Angle of attack)}, \quad q \text{ (Pitch rate)}$$
$$\delta_E \text{ (Elevator deflection)}, \quad \theta \text{ (Pitch)}.$$



(a) Side view

State-space equation: $\dot{x} = Ax + B\delta_E$

$$x = \begin{bmatrix} \Delta u \\ \alpha \\ q \\ \theta \end{bmatrix}, \quad A = \begin{bmatrix} -.0507 & -3.861 & 0 & -32.2 \\ -.00117 & -.5164 & 1 & 0 \\ -.000129 & 1.4168 & -.4932 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ .0717 \\ -.165 \\ 0 \end{bmatrix}.$$

Stable (trimmed) flight is subject to $x(t) \in K$, where $K = S\mathbb{R}^4_+$ is some simplicial cone.

Goal: Choose δ_E so that $\exists\, t_0 \geq 0$ and $\forall\, t \geq t_0$, $x(t) \in K$.

Do change of variables $y(t) = S^{-1}x$ and pursue **feedback** law $\delta_E = Fy(t)$ so that $\tilde{A} + \tilde{B}F$ has Perron-Frobenius property.

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- “irreducibility” ?

Counterexample 1

The Perron-Frobenius property for A **does not** imply eventual nonnegativity of A. E.g., consider

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 6 & 2 & -4 \\ 3 & 1 & 0 \end{bmatrix}.$$

- A has Perron-Frobenius property ($\rho(A) = 4$, $x = [1\ 1\ 1]^T$).
- A^T has Perron-Frobenius property ($\rho(A^T) = 4$, $x = [2\ 1\ 0]^T$).
- Third column of A^k has a negative entry $\forall\, k \geq 2$, i.e., A is <u>not eventually nonnegative</u>.

Counterexample 2

A non-nilpotent eventually nonnegative matrix **need not** have the strong Perron-Frobenius property; e.g.,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}, \quad A^k = 2^k \begin{bmatrix} 1 & 1 & k & k \\ 1 & 1 & k & k \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \; (k \geq 2).$$

A has $\rho(A) = 2$ as an eigenvalue of multiplicity 2.
(Crucial that $\mathrm{index}_0(A) = 2$: size of largest Jordan block of 0.)

Some of many challenges

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- Perron-Frobenius property + ? $\Longleftrightarrow$ Ev. Exp. nonnegativity
- When is PAQ eventually nonnegative (positive)?

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Let A be ev. nonnegative (positive) and D nonn. diag. matrix

Some of many challenges

- Perron-Frobenius property + ? $\Longrightarrow$ Eventual nonnegativity
- Perron-Frobenius property + ? $\Longleftrightarrow$ Eventual nonnegativity
- Perron-Frobenius property + ? $\Longleftrightarrow$ Ev. Exp. nonnegativity
- When is PAQ eventually nonnegative (positive)?
- (When) is $\rho(A)$ a non-decreasing function of the entries?

Let A be ev. nonnegative (positive) and D nonn. diag. matrix

- When is $A + D$ eventually nonnegative (positive)?

Some of many challenges

- Perron-Frobenius property + ? $\Longrightarrow$ Eventual nonnegativity
- Perron-Frobenius property + ? $\Longleftrightarrow$ Eventual nonnegativity
- Perron-Frobenius property + ? $\Longleftrightarrow$ Ev. Exp. nonnegativity
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Let A be ev. nonnegative (positive) and D nonn. diag. matrix

- When is $A + D$ eventually nonnegative (positive)?
- When is DA eventually nonnegative (positive)?

Comments...

- For qualitative study of eventual nonnegativity, look at papers in ELA Vol. 19 (AIM Volume)

- For M-matrix types look at [Elhashash, Szyld, LAA & ELA 2008] and [Olesky, van den Driessche, T, ELA 2009]

- This presentation (will soon be) available at
  www.math.wsu.edu/faculty/tsat/

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