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ON GENERALIZED EVEN AND ODD OSCILLATORY OPERATORS

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Definitions

Definition 1

Let A be a linear operator acting in the space $\mathbb{R}^n$. In this case we can define operators $\otimes^j A$ and $\wedge^j A$ $(j = 1, \ldots, n)$, i.e. the j-th tensor and the j-th exterior power of the operator A. They acts, respectively, in the space $\otimes^j \mathbb{R}^n = \mathbb{R}^{n^j}$ and $\wedge^j \mathbb{R}^n = \mathbb{R}^{C_n^j}$.

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If $e_1, \ldots, e_n$ is a basis in $\mathbb{R}^n$, then all the possible exterior products of the form $e_{i_1} \wedge \ldots \wedge e_{i_j}$, where $1 \leq i_1 < \ldots < i_j \leq n$, form a basis in the jth exterior power $\wedge^j \mathbb{R}^n$ of the space $\mathbb{R}^n$.

Theorem

Let $\{\lambda_i\}_{i=1}^{n}$ be all eigenvalues of the operator A, repeated according to multiplicity. Then all the possible products of the type $\{\lambda_{i_1} \dots \lambda_{i_j}\}$, where $1 \leq i_1 < \dots < i_j \leq n$, form all the possible eigenvalues of the exterior power $\wedge^j A$, repeated according to multiplicity [3].

- Let a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ be defined by a $n \times n$ matrix $\mathbf{A}$ in the basis $e_1, \ \ldots, \ e_n$. Then the matrix of its jth exterior power $\wedge^j A$ in the basis, which consists of all the possible exterior products of the form $e_{i_1} \wedge \ldots \wedge e_{i_j} \quad (1 \leq i_1 < \ldots < i_j \leq n)$, coincides with the jth compound matrix $\mathbf{A}^{(j)}$ of the initial matrix $\mathbf{A}$. (Here the jth compound matrix $\mathbf{A}^{(j)}$ is a $C_n^j \times C_n^j$ matrix, which consists of all the minors of the jth order of the initial matrix $\mathbf{A}$. The minors are numerated in the lexicographic order.)

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- In the case, when the operator A is defined by its matrix, the statement about the eigenvalues of its jth exterior power $\wedge^j A$ turns into the Kronecker theorem (see [1], p. 80, theorem 23) about the eigenvalues of the jth compound matrix. The proof of the Kronecker theorem without using exterior products is given in monograph [1].

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K–primitive

A linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is called *K–primitive*, if there exists a proper cone K, such that $AK \subseteq K$ and the only nonempty subset of $\partial(K)$ which is left invariant by A is $\{0\}$.

Generalized Oscillatory operators

Generalized Oscillatory

A linear operator A is called *generalized oscillatory* if it is K-primitive with respect to a proper cone $K_1 \subset \mathbb{R}^n$, and for every j $(1 < j \leq n)$ its j-th exterior power $\wedge^j A$ is K-primitive with respect to a proper cone $K_j \subset \mathbb{R}^{C_n^j}$.

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Generalized Even (Odd) Oscillatory

A linear operator A is called *generalized even (odd) oscillatory* if for every even (respectively odd) j $(1 \leq j \leq n)$ its j-th exterior power $\wedge^j A$ is K-primitive with respect to a proper cone $K_j \subset \mathbb{R}^{C_n^j}$.

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K–primitive

If the matrix **A** of a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is primitive, then A is K-primitive with respect to the cone K_+ of all nonnegative vectors from the space $\mathbb{R}^n$. The statement, that if the matrix **A** is similar to a primitive matrix, then the corresponding operator A is K-primitive with respect to some polyhedral cone K in $\mathbb{R}^n$, easily follows from the above reasoning. In some special cases we can see, if the matrix **A** is similar to a primitive matrix, just looking at its structure.

Theorem 1

Let a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ *be generalized oscillatory. Then all the eigenvalues of the operator* A *are simple, positive and different in modulus from each other:*

$$\rho(A) = \lambda_1 > \lambda_2 > \ldots > \lambda_n > 0.$$

Theorem 2

Let a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ be even generalized oscillatory. Then the algebraic multiplicity $m(\lambda)$ of any eigenvalue λ of the operator A is not greater than 2. The following inequalities for the modules of the eigenvalues are true:

$$\rho(A) = |\lambda_1| \leq |\lambda_2| < |\lambda_3| \leq |\lambda_4| < \ldots .$$

(The eigenvalues of A are repeated according to multiplicity in the above numeration.) Moreover, for every pair $\lambda_i \lambda_{i+1}$ $(i = 1,\ 3,\ 5,\ \ldots)$ the following equality is true: $\arg(\lambda_{i+1}) = -arg(\lambda_i)$. If n is odd, then λ_n is real.

Theorem 3

Let a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ be odd generalized oscillatory. Then the algebraic multiplicity $m(\lambda)$ of any eigenvalue λ of the operator A is not greater than 2. The following inequalities for the modules of the eigenvalues are true:

$$\rho(A) = |\lambda_1| < |\lambda_2| \leq |\lambda_3| < |\lambda_4| \leq \dots .$$

(The eigenvalues of A are repeated according to multiplicity in the above numeration.) Moreover, $\lambda_1 = \rho(A)$ is a simple positive eigenvalue of A. If n is even, then λ_n is real. For every pair $\lambda_i \lambda_{i+1}$ $(i = 2, 4, 6, \dots)$ the following equality is true: $\arg(\lambda_{i+1}) = -arg(\lambda_i)$.

Compound Matrix

Let **A** be an $n \times n$ matrix, then the jth compound matrix $\mathbf{A}^{(j)}$of the matrix **A** is defined as the matrix of order $C_n^j \times C_n^j$, which consists of all the minors of the jth order of the initial matrix **A**. The minors are numerated in the lexicographic order.

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For Example: If

$$\mathbf{A} = \begin{pmatrix} -3 & -1 & 1 & 2 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 0 & 0 & -2 \end{pmatrix}.$$

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$$\mathbf{A}^{(2)} = \begin{pmatrix} 4 & -10 & -5 & -2 & 1 & -5 \\ -1 & -5 & -7 & -2 & -3 & -1 \\ 3 & -3 & -3 & 0 & 1 & -1 \\ 3 & -5 & -1 & -4 & -2 & 2 \\ 3 & 9 & -4 & 0 & 1 & -3 \\ -3 & -3 & -5 & 0 & -1 & -1 \end{pmatrix}.$$

$\mathcal{J}$–sign-symmetric Matrix

$\mathcal{J}$–sign-symmetric

A matrix **A** of a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is called $\mathcal{J}$*–sign-symmetric*, if there exists such a subset $\mathcal{J} \subseteq \{1, \ldots, n\}$, that both the conditions (a) and (b) are true:

(a) the inequality $a_{ij} \leq 0$ follows from the inclusions $i \in \mathcal{J}$, $j \in \{1, \ldots, n\} \setminus \mathcal{J}$ and from the inclusions $j \in \mathcal{J}$, $i \in \{1, \ldots, n\} \setminus \mathcal{J}$ for any two numbers i, j;

(b) one of the inclusions $i \in \mathcal{J}$, $j \in \{1, \ldots, n\} \setminus \mathcal{J}$ or $j \in \mathcal{J}$, $i \in \{1, \ldots, n\} \setminus \mathcal{J}$ follows from the strict inequality $a_{ij} < 0$.

strictly $\mathcal{J}$–sign-symmetric Matrix

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A matrix $\mathbf{A}$ is called *strictly $\mathcal{J}$–sign-symmetric*, if $\mathbf{A}$ does not contain zero elements and there exists such a subset $\mathcal{J} \subseteq \{1, \ldots, n\}$, that the inequality $a_{ij} < 0$ is true if and only if one of the numbers i, j belongs to the set $\mathcal{J}$, and the other belongs to the set $\{1, \ldots, n\} \setminus \mathcal{J}$.

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$$\mathbf{A}^{(2)} = \begin{pmatrix} 149 & -33 & -56 & -60 & -156 & 12 \\ -33 & 21 & 12 & 32 & 34 & -10 \\ -56 & 12 & 44 & 22 & 90 & -2 \\ -60 & 32 & 22 & 52 & 62 & -14 \\ -156 & 34 & 90 & 62 & 210 & -10 \\ 12 & -10 & -2 & -14 & -10 & 6 \end{pmatrix}.$$

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where the set $\mathcal{J}$ is equal to $\{1, 6\}$ or $\{2, 3, 4, 5\}$.

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$$\mathbf{A}^{(2)} = \begin{pmatrix} 57 & 38 & 15 & -19 & -48 & -27 \\ 35 & 24 & 9 & -10 & -30 & -18 \\ 9 & 6 & 3 & -3 & -6 & -3 \\ -33 & -21 & -9 & 23 & 24 & 9 \\ -72 & -48 & -18 & 24 & 72 & 42 \\ -39 & -27 & -9 & 9 & 42 & 27 \end{pmatrix}.$$

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where the set $\mathcal{J}$ is equal to $\{1, 2, 3\}$ or $\{4, 5, 6\}$.

$\mathcal{J}$–sign-symmetric primitive Matrix

- If the matrix **A** is $\mathcal{J}$–sign-symmetric (strictly $\mathcal{J}$–sign-symmetric), then it is similar to some nonnegative (respectively positive) matrix. Moreover, the matrix of the similarity transformation is diagonal, and its diagonal elements are equal to ± 1.

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- If the matrix **A** is $\mathcal{J}$–sign-symmetric (strictly $\mathcal{J}$–sign-symmetric), then it is similar to some nonnegative (respectively positive) matrix. Moreover, the matrix of the similarity transformation is diagonal, and its diagonal elements are equal to ± 1.
- It's easy to see, that if the matrix **A** is $\mathcal{J}$–sign-symmetric, and the matrix $\mathbf{A}^m$ is strictly $\mathcal{J}$–sign-symmetric for some natural number m, then the matrix **A** is similar to some primitive matrix with the diagonal matrix of the similarity transformation. Let us call such matrices *$\mathcal{J}$–sign-symmetric primitive*. In this case the linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$, defined by the matrix **A**, is K-primitive with respect to some cone spanned on the vectors $e'_1, \ \ldots, \ e'_n$, where each vector e'_i is equal either to e_i or to $-e_i$ $\ (i = 1, \ \ldots, \ n)$.

Theorem 1$'$

Let the matrix $\mathbf{A}$ *of a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be* $\mathcal{J}$*–sign-symmetric primitive, and let the* j*th compound matrix* $\mathbf{A}^{(j)}$ *be also* $\mathcal{J}$*–sign-symmetric primitive for every* j $(1 < j \leq n)$*. Then all the eigenvalues of the operator* A *are simple, positive and different in modulus from each other:*

$$\rho(A) = \lambda_1 > \lambda_2 > \ldots > \lambda_n > 0.$$

Theorem 2′

Let the jth compound matrix $\mathbf{A}^{(j)}$ *of the matrix* $\mathbf{A}$ *of a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be* $\mathcal{J}$*–sign-symmetric primitive for every even* j $(1 \leq j \leq n)$*. Then the algebraic multiplicity* $m(\lambda)$ *of any eigenvalue* λ *of the operator* A *is not greater than* 2*. The following inequalities for the modules of the eigenvalues are true:*

$$\rho(A) = |\lambda_1| \leq |\lambda_2| < |\lambda_3| \leq |\lambda_4| < \ldots .$$

(The eigenvalues of A *are repeated according to multiplicity in the above numeration.) Moreover, for every pair* $\lambda_i\lambda_{i+1}$ $(i = 1,\ 3,\ 5,\ \ldots)$ *the following equality is true:* $\arg(\lambda_{i+1}) = -arg(\lambda_i)$*. If* n *is odd, then* λ_n *is real.*

Theorem 3′

Let the jth compound matrix $\mathbf{A}^{(j)}$ *of the matrix* $\mathbf{A}$ *of a linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be* $\mathcal{J}$*–sign-symmetric primitive for every odd* j $(1 \leq j \leq n)$*. Then the algebraic multiplicity* $m(\lambda)$ *of any eigenvalue* λ *of the operator* A *is not greater than* 2*. The following inequalities for the modules of the eigenvalues are true:*

$$\rho(A) = |\lambda_1| < |\lambda_2| \leq |\lambda_3| < |\lambda_4| \leq \dots .$$

(The eigenvalues of A *are repeated according to multiplicity in the above numeration.) Moreover,* $\lambda_1 = \rho(A)$ *is a simple positive eigenvalue of* A*. If* n *is even, then* λ_n *is real. For every pair* $\lambda_i \lambda_{i+1}$ $(i = 2,\ 4,\ 6,\ \dots)$ *the following equality is true:* $\arg(\lambda_{i+1}) = -arg(\lambda_i)$.

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