Matrix Version of the Chebyshev and Kantorovich Inequalities

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Matrix inequalities arise in various branches of mathematics and science such as system and control theory [Boyd *et. al* (1994)] and optimization [Todd (2001)]. Matrix inequalities are also important tools in quantum statistical inference and quantum information theory [Barndorff-Neilsen (2003), Nielsen (1999)].

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{R}^n$, $\mathbb{C}^n$	Set of n tuples with components from $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{R}^n_+$	Set of n tuples of positive real numbers
$\mathcal{M}_n$	Algebra of complex matrices of order n
I_n	Identity matrix in $\mathcal{M}_n$
O_n	Zero matrix in $\mathcal{M}_n$
$\mathcal{H}_n$	Set of all Hermitian matrices in $\mathcal{M}_n$

Definition 1.

$A \in \mathcal{H}_n$ is *positive semidefinite (positive definite)* if and only if all of its eigenvalues are *nonnegative (positive)*.

$\mathcal{S}_n$	Set of all positive semidefinite matrices in $\mathcal{H}_n$
$\mathcal{P}_n$	Set of all positive definite matrices in $\mathcal{H}_n$.

For $A, B \in \mathcal{H}_n$,

$B \geq A$	means	$B - A \in \mathcal{S}_n$
$B > A$	means	$B - A \in \mathcal{P}_n$

Let I be a interval.

$\mathcal{H}_n(I)$ Set of all Hermitian matrices in $\mathcal{M}_n$ whose spectrum is contained in I.

Let $f : I \to \mathbb{R}$ be a function. Every element A in $\mathcal{H}_n$ admits a *spectral decomposition*

$$A = \sum_{j=1}^{n'} \lambda_j P_j \ (n' \leq n)$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_{n'}\}$ is the spectrum of A and $P_1, P_2, \ldots, P_{n'}$ are orthogonal projections such that $\sum_{j=1}^{n'} P_j = I_n$.

If $A \in \mathcal{H}_n(I)$, we define

$$f(A) = \sum_{j=1}^{n'} f(\lambda_j) P_j \quad (n' \leq n).$$

The matrix $f(A)$ is an element of $\mathcal{H}_n$.

Let $A = (a_{ij}) \in \mathcal{M}_n$ and $B = (b_{ij}) \in \mathcal{M}_n$. The *tensor product*, $A \otimes B$, is defined to be the block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix} \in \mathcal{M}_{n^2}. \tag{1}$$

The *Hadamard product*, $A \circ B$, is defined by

$$A \circ B = (a_{ij}b_{ij}) \in M_n.$$

Some Inequalities Involving Hadamard Product

The inequality

$$\left(\sum_{j=1}^{m} w_j a_j\right)\left(\sum_{j=1}^{m} w_j b_j\right) \leq \left(\sum_{j=1}^{m} w_j\right)\left(\sum_{j=1}^{m} w_j a_j b_j\right) \tag{2}$$

holds for all $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0,\ b_1 \geq b_2 \geq \cdots \geq b_m \geq 0$ and weights $w_j \geq 0,\ 1 \leq j \leq m$. Hardy, Littlewood and Polya, [Hardy et al. (1959), page 43], attribute this inequality to Chebyshev.

Theorem 2.

Let $A_j, B_j \in \mathcal{P}_n$, $1 \leq j \leq m$ be such that $A_1 \geq \cdots \geq A_m \geq O_n$ and $B_1 \geq \cdots \geq B_m \geq O_n$. Then

$$\left(\sum_{j=1}^{m} w_j A_j\right) \circ \left(\sum_{j=1}^{m} w_j B_j\right) \leq \left(\sum_{j=1}^{m} w_j\right) \left(\sum_{j=1}^{m} w_j (A_j \circ B_j)\right)$$

where $w_j \geq 0$, $1 \leq j \leq m$, are weights.

If $0 < a \leq a_j \leq b,\ w_j \geq 0,\ 1 \leq j \leq m$, Kantorovich's inequality states that

$$\left(\sum_{j=1}^{m} w_j a_j\right)\left(\sum_{j=1}^{m} \frac{w_j}{a_j}\right) \leq \frac{(a+b)^2}{4ab}\left(\sum_{j=1}^{m} w_j\right)^2. \tag{3}$$

We state and prove matrix version of inequality (3) involving the Hadamard product.

Our next result is a Hadamard product version of inequality (3).

Theorem 3.

Let $A_j \in \mathcal{P}_n$, $1 \le j \le m$, *be such that* $O_n < aI_n \le A_j \le bI_n$, $1 \le j \le m$, $a, b > 0$. *Then*

$$\left(\sum_{j=1}^{m} W_j^{1/2} A_j W_j^{1/2}\right) \circ \left(\sum_{j=1}^{m} W_j^{1/2} A_j^{-1} W_j^{1/2}\right) \le \frac{a^2+b^2}{2ab}\left(\sum_{j=1}^{m} W_j\right) \circ \left(\sum_{j=1}^{m} W_j\right)$$

for all $W_j \in \mathcal{S}_n$, $1 \le j \le m$.

As a consequence of the above inequality we will deduce the following corollary.

Corollary 4.

Let $A_j \in \mathcal{P}_n$ *be such that* $O_n < aI_n \leq A_j \leq bI_n, \ a, b > 0,$ *and* $w_j \geq 0, \ 1 \leq j \leq m,$ *be weights. Then*

$$\left(\sum_{j=1}^{m} w_j A_j\right) \circ \left(\sum_{j=1}^{m} w_j A_j^{-1}\right) \leq \left(\frac{a^2 + b^2}{2ab}\right) \left(\sum_{j=1}^{m} w_j\right)^2 I_n.$$

Remarks

The example $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $a = \dfrac{3-\sqrt{5}}{2}$, $b = \dfrac{3+\sqrt{5}}{2}$ shows that the inequality

$$A \circ A^{-1} \leq \frac{(a+b)^2}{4ab} I_2$$

need not be true.

Let $f(t) = \left(\sum_{j=1}^{m} w_j a_j^t\right)\left(\sum_{j=1}^{m} w_j a_j^{-t}\right)$, $w_j \geq 0$, $a_j > 0$, $1 \leq j \leq m$, be a function defined on $[0, \infty)$. Then f is monotone increasing function. We will prove that when real numbers w_j, a_j, $1 \leq j \leq m$, are replaced with matrices, the following result holds.

Theorem 5.

Let $0 \leq \alpha < \beta$. Then

$$\left(\sum_{j=1}^{m} W_j^{1/2} A_j^{\alpha} W_j^{1/2}\right) \circ \left(\sum_{j=1}^{m} W_j^{1/2} A_j^{-\alpha} W_j^{1/2}\right) \leq \left(\sum_{j=1}^{m} W_j^{1/2} A_j^{\beta} W_j^{1/2}\right) \circ \left(\sum_{j=1}^{m} W_j^{1/2} A_j^{-\beta} W_j^{1/2}\right)$$

for all $A_j \in \mathcal{P}_n$ and $W_j \in \mathcal{S}_n$, $1 \leq j \leq m$.

As a consequence of the above theorem a generalization of Fiedler's inequality $A \circ A^{-1} \geq I_n, \ A \in \mathcal{P}_n$, is obtained as a corollary. It is mentioned that there are several generalizations of Kantorovich and Fiedler's inequality; [Baksalary and Puntanen (1991), Bapat and Kwong (1987), Marshal and Olkin (1990), Singh et al. (2000), see].

Next we will prove the following convexity theorem involving Hadamard product.

Theorem 6.

The function

$$f(t) = A^{1+t} \circ B^{1-t} + A^{1-t} \circ B^{1+t}$$

is convex on the interval [-1,1] and attains its minimum at $t = 0$ *for all* $A, B \in \mathcal{P}_n$.

Corollary 7.

The function

$$g(t) = A^t \circ B^{1-t} + A^{1-t} \circ B^t$$

is decreasing on [0,1/2], increasing on [1/2,1], and attains its minimum at $t = \dfrac{1}{2}$ *for all* $A, B > 0$.

As a corollary to this theorem we obtain the following result which may be regarded as a generalization of arithmetic-geometric mean inequality for numbers.

Corollary 8.

Let $0 \leq t \leq 1$. *Then,*

$$2|||A^{1/2} \circ B^{1/2}||| \leq |||A^t \circ B^{1-t} + A^{1-t} \circ B^t||| \leq |||A + B|||$$

for all unitarily invariant norms $|||\cdot|||$ *and all* $A, B \in \mathcal{P}_n$.

This corollary is a refinement of Theorem 4.4 in [Aujla and Vasudeva(1995)].

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