Classes of General $H-$ and non$-H-$Matrices*

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Outline of talk:

A. INTRODUCTION

(Nonsingular) $H-$Matrices

(Published Material

jointly done with Maria Alanelli)

B. MAIN TALK

General $H-$Matrices

(Unpublished Material

jointly done with Rafael Bru and Isabel Giménez)

1. Introduction

Definition 1. *Given*

$$A \in \mathbb{C}^{n,n} \quad \textit{and} \quad N := \{1, 2, \cdots, n\} \tag{1}$$

the Comparison Matrix of A is defined by

$$\mathcal{M}(A) \ := \ \begin{cases} m_{ii} = |a_{ii}| & \textit{for all} \ \ i \in N \\ m_{ij} = -|a_{ij}| & \textit{for all} \ \ i \neq j \in N \end{cases} \tag{2}$$

A. M. Ostrowski (Comment. Math. Helv. (1937)):

Definition 2. *A matrix $A \in \mathbb{C}^{n,n}$, with $a_{ii} \neq 0, i = 1(1)n$, is defined to be a (nonsingular) $H-$matrix iff the Jacobi iteration matrix associated with its comparison matrix converges. That is*

$$\rho\Big(I - (diag(\mathcal{M}(A)))^{-1}\,\mathcal{M}(A)\Big) \ < \ 1. \tag{3}$$

(There are many Definitions equivalent to the above one.)

- DIRECT:

  Y.-M. Gao/X.-H. Wang (LAA (1992),(1996))
  T.-Z. Huang (LAA (1995))
  T.-B. Gan/T.-Z. Huang (LAA (2003))
  Lj. Cvetković/V. Kostić (JCAM (2005)) etc.

- ITERATIVE:

  M. Harada/M. Usui/H. Niki (IJCM (1996))
  B. Li/ L. Li/M. Harada/H. Niki/M.J. Tsatsomeros (LAA (1998))
  L. Li (SIMAX (2002))
  K. Ojiro/H. Niki/M. Usui (JCAM (2003))
  A.H. (LAA (2004))

(Although **not** quite clear) All Iterative Algorithms applied the **Power Method** on Nonnegative Matrices

2. The Algorithm $\mathbb{H}$ ($\mathbb{AH}$): M. Alanelli/A.H. (SIMAX (2006))

INPUT: An **irreducible** matrix $A := [a_{ij}] \in \mathbb{C}^{n,n}$.

1. If $a_{ii} = 0$ for some $i \in \mathbb{N}$, "A is **not** an $H-$matrix", STOP; Otherwise
2. Set $D = I$, $A^{(0)} = (\text{diag}(A))^{-1} A$, $D^{(0)} = I$, $k = 1$
3. Compute $D = DD^{(k-1)}$, $A^{(k)} = \left(D^{(k-1)}\right)^{-1} A^{(k-1)} D^{(k-1)} = [a_{ij}^{(k)}]$
4. Compute $s_i^{(k)} = \sum_{j=1,\ j\neq i}^{n} |a_{ij}^{(k)}|$, $i = 1(1)n$,

$$s^{(k)} = \min_{i=1(1)n} s_i^{(k)},\ S^{(k)} = \max_{i=1(1)n} s_i^{(k)}$$

5. If $s^{(k)} > 1$, "A is **not** an $H-$matrix", STOP; Otherwise
6. If $S^{(k)} < 1$, "A **is** an $H-$matrix", STOP; Otherwise
7. If $S^{(k)} = s^{(k)}$, "$\mathcal{M}(A)$ is **singular**", STOP; Otherwise
8. Set $d = [d_i]$, where $d_i = \frac{1+s_i^{(k)}}{1+S^{(k)}}$, $i = 1(1)n$
9. Set $D^{(k)} =$ diag(d), $k = k + 1$; Go to Step 3.

<u>Statement of Main Theorem</u>

Lemma 1. *For any given* **irreducible** *matrix* $A \in \mathbb{C}^{n,n}$ *Algorithm* $\mathbb{AH}$ *always terminates in finite iteration steps.*

<u>Auxiliary Statement</u>

Theorem 1. *For any given irreducible matrix* $A \in \mathbb{R}^{n,n}$, $A \geq 0$, *let* P^* *be the* **hyperoctant** *of vectors* $x > 0$. *Then, for any* $x \in P^*$, *either*

$$\min_{i=1(1)n} \left\{ \frac{(Ax)_i}{x_i} \right\} < \rho(A) < \max_{i=1(1)n} \left\{ \frac{(Ax)_i}{x_i} \right\}$$

or

$$\frac{(Ax)_i}{x_i} = \rho(A), \;\; i = 1(1)n.$$

Use of the quotient appearing in Theorem 1 $\left(x_i^{(k+1)} = \frac{(Ax^{(k)})_i}{x_i^{(k)}} \right)$, in an iterative manner, leads to the Power Method for the matrix A. (Exercise 2.2, p. 39 in Varga's Book.)

3. The Algorithm $\mathbb{H}\mathbf{2}$ ($\mathbb{A}\mathbb{H}\mathbf{2}$): M. Alanelli/A.H. (LAA (2008))

The Irreducible / Reducible Case

U. G. Rothblum (LAA (1975))

H. Schneider (LAA (1986))

R. Bru/M. Neumann (LAA (1988))

A. Bermann/R. J. Plemmons (1994)

Lemma 2. *Let* $A \geq 0$, $A \in \mathbb{R}^{n,n}$, *be reducible. There exists a permutation matrix* P *that puts* A *into its* **Frobenius normal form**

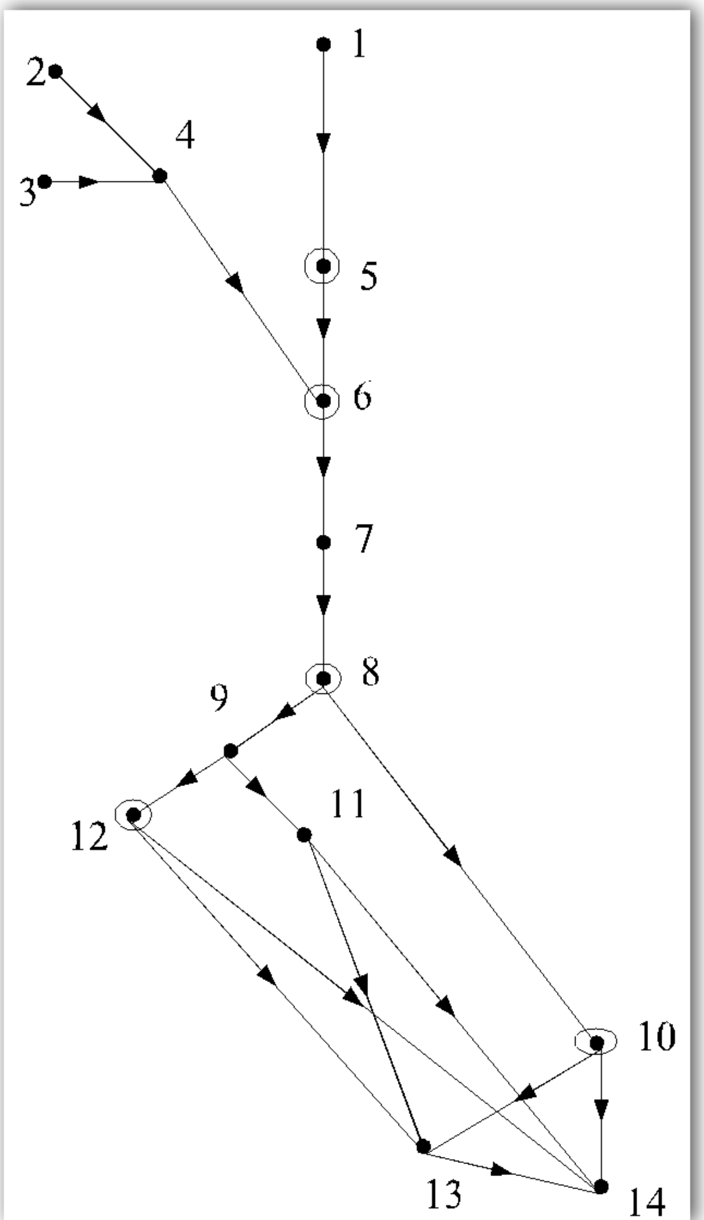
$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1,p-1} & A_{1p} \\ & A_{22} & \cdots & \cdots & A_{2,p-1} & A_{2p} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \vdots & \vdots \\ & & & & A_{p-1,p-1} & A_{p-1,p} \\ & & & & & A_{pp} \end{bmatrix}, \qquad (4)$$

where $A_{ii} \in \mathbb{R}^{n_i,n_i}$, $i = 1(1)p$, $\sum_{i=1}^{p} n_i = n$, *is irreducible or a* 1×1 *nullmatrix.*

<u>Material from Combinatorial/Graph Matrix Theory</u>

Definition 3. *Let* $A \geq 0$, $A \in \mathbb{R}^{n,n}$. *We define the* **block (directed) graph** *of A*, $G(A)$, *to be the graph with vertices (nodes)* $1(1)p$, *where an edge (arc) leads from i to j if and only if* $A_{ij} \neq O$.

$$
\begin{bmatrix}
A_{11} & 0 & 0 & 0 & A_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{22} & 0 & A_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{33} & A_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{44} & 0 & A_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{A}_{55} & A_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{A}_{66} & A_{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{77} & A_{78} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_{88} & A_{89} & A_{8,10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{99} & 0 & A_{9,11} & A_{9,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_{10,10} & 0 & 0 & A_{10,13} & A_{10,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{11,11} & 0 & A_{11,13} & A_{11,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_{12,12} & A_{12,13} & A_{12,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{A}_{13,13} & A_{13,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{A}_{14,14}
\end{bmatrix}
$$



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Theorem 2. *If the graph of A, $G(A)$, does not consist of a union of disjoint subgraphs, then we apply a permutation transformation on A, let it be QAQ^T, keeping its Frobenius normal form, so that its basic classes are in increasing order of their heights into larger principal blocks as final classes. Then, the application of Algorithm $\mathbb{AH}$ (actually the Power Method) to the new form of A makes all row sums of the major principal blocks corresponding to basic classes tend to $\rho(A)$ in the limit. If there is a last major principal block of nonbasic classes that do not have access to any basic one, let it be $\widetilde{A}$, then, by following the previous described rules and depending on $G(\widetilde{A})$, the application of Algorithm $\mathbb{AH}$ makes the row sums of $\widetilde{A}$ tend to limits that are strictly less than $\rho(A)$.*

$$\rho(A) = \rho(A_{55}) = \rho(A_{66}) = \rho(A_{88}) = \rho(A_{10,10}) = \rho(A_{12,12}) > \rho(A_{13,13}) > \rho(A_{14,14}),$$
$$\rho(A_{13,13}) > \rho(A_{11,11})$$

$$\left[\begin{array}{cc|cccc|cc|ccc|ccc}
A_{11} & A_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{A}_{55} & 0 & 0 & 0 & A_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & A_{22} & 0 & A_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{33} & A_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{44} & A_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{A}_{66} & A_{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & A_{77} & A_{78} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_{88} & A_{8,10} & A_{89} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_{10,10} & 0 & 0 & 0 & A_{10,13} & A_{10,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{99} & A_{9,12} & A_{9,11} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{A}_{12,12} & 0 & A_{12,13} & A_{12,14} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{11,11} & A_{11,13} & A_{11,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{13,13} & A_{13,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{14,14}
\end{array}\right]$$

Algorithm $\mathbb{AH}2$.

INPUT: The matrix $A := [a_{ij}] \in \mathbb{C}^{n,n}$, and the maximum number of iterations allowed ("maxiter")
OUTPUT: $D = D^{(0)}D^{(1)} \cdots D^{(k)} \in \mathfrak{D}_{D^{-1}A} \equiv \mathfrak{D}_A \quad \notin \mathfrak{D}_A$ if A **is** or is **not** an $H-$matrix, respectively
1. If $a_{ii} = 0$ for some $i \in \mathbb{N}$, "A is **not** an $H-$matrix", STOP; Otherwise
2. Set $D = I$, $A^{(0)} = (\text{diag}(A))^{-1} A$, $D^{(0)} = I$, $k = 1$
3. Compute $D = DD^{(k-1)}$, $A^{(k)} = \left(D^{(k-1)}\right)^{-1} A^{(k-1)} D^{(k-1)} = [a_{ij}^{(k)}]$
4. Compute $s_i^{(k)}$, $i = 1(1)n$, $s^{(k)} = \min_{i=1(1)n} s_i^{(k)}$, $S^{(k)} = \max_{i=1(1)n} s_i^{(k)}$
5. If $s^{(k)} > 1$, "A is **not** an $H-$matrix", STOP; Otherwise
6. If $S^{(k)} < 1$, " A **is** an $H-$matrix", STOP; Otherwise
7. If $S^{(k)} = s^{(k)}$, " $\mathcal{M}(A)$ is **singular**", STOP; Otherwise
8. Set $d = [d_i]$, where

$$d_i = \frac{1 + s_i^{(k)}}{1 + S^{(k)}}, \; i = 1(1)n$$

9. Set $D^{(k)} = \text{diag}(d)$, If $k <$ maxiter, $k = k + 1$, Go to Step 3; Otherwise

10. Find $\mathbb{N}_0^{(\text{iter})}$ and $n_0^{(\text{iter})}$
11. If $n_0^{(\text{iter})} = 1$, "Inconclusive, increase maxiter", STOP; Otherwise
12. Compute

$$s_{i_j}^{(\text{iter})} = \sum_{l=1,\ l\neq j}^{n_0^{(\text{iter})}} |a_{i_j,i_l}^{(\text{iter})}|, \quad j = 1(1)n_0^{(\text{iter})}, \quad i_j, i_l \in \mathbb{N}_0^{(\text{iter})}$$

13. If $s_{i_j}^{(\text{iter})} \geq 1$, $j = 1(1)n_0^{(\text{iter})}$, $i_j \in \mathbb{N}_0^{(\text{iter})}$, "$A$ is **not** an $H-$matrix", STOP; Otherwise
14. Update $\mathbb{N}_0^{(\text{iter})}$ (by deleting $i_j \in \mathbb{N}_0^{(\text{iter})} : s_{i_j} < 1$) and $n_0^{(\text{iter})}$; Go to Step 11.

4. Identifying Classes of General $H-$ and non$-H-$Matrices

R. Bru/I. Giménez/A.H.
(unpublished material)

The class of $H-$matrices was extended to *General $H-$matrices:*
H. Schneider (Proc. Edim. Math. Soc. (1956))

Definition 4. *A matrix $A \in \mathbb{C}^{n,n}$ is a general $H-$matrix iff $\mathcal{M}(A)$ written uniquely as*

$$\mathcal{M}(A) \; := \; sI - B \;\; \textit{with} \;\; s := \max_{i \in N} |a_{ii}| \;\; \textit{and} \;\; B \geq O \quad \Longrightarrow \quad s \geq \rho(B). \tag{5}$$

R. Bru/C. Corral/I. Giménez/J. Mas (LAA (2008))

$A \in \mathbb{C}^{n,n}, \quad \mathcal{M}(A) = sI - B, \quad s = \max_{i=1(1)n} \lvert a_{ii} \rvert, \ B \geq 0$			
$H-$*matrices* ($\mathcal{H}$)			*non*$-H-$*matrices* ($_N\mathcal{H}$)
$s > \rho(B)$	$s = \rho(B)$		$s < \rho(B)$
$\exists\ J_A\ (A \in \mathcal{H}_I)$	$\nexists\ J_A\ (A \in \mathcal{H}_S)$	$\exists\ J_A\ (A \in \mathcal{H}_M)$	$\nexists$ *or* $\exists\ J_A\ (A \in {}_N\mathcal{H})$
A *"invertible"*	A *"singular"*	A *"mixed"*	
irreducible or reducible	$---$ *reducible*	*irreducible or reducible*	*irreducible or reducible*

PROBLEM: Given $A \in \mathbb{C}^{n,n}$ identify its general $H-$matrix character and the class to which it belongs.

NOTE: If the matrix is reducible consider ONLY the **diagonal blocks** of the **Frobenius normal form** (**bdFnf**).

The NEW Algorithm will be split into three PARTS.

1) Given $A \in \mathbb{C}^{n,n}$, identify the **irreducible** or **reducible** character of A.

2) If A is **reducible** find a **block permutation** of the **diagonal blocks** (**bdFnf**) of a **Frobenius normal form** (**Fnf**) of A $(p! \prod_{i=1}^{p}(n_i!))$

3) If necessary, apply a **slight modification** of **AH** to one or more blocks of the **Frobenius normal form** of A until the identification is made

PART 1: Irreducible or Reducible?

Lemma (see, e.g., Varga (2000)): If $A \in \mathbb{R}^{n,n}$ is irreducible and $A \geq O$, then $(I + A)^{n-1} > O$.

Theorem 3. *A matrix $A \in \mathbb{C}^{n,n}$ is irreducible iff*

$$(I + |A|)^{n-1} > O$$

Notes:

i) Begin with $\underline{\underline{C = I + |A|}}$ and stop either as soon as the number of nonzero elements (**nnz**) of C^k, $k = 1(1)n-1$, is n^2, or **nnz**(C^k) =**nnz**(C^{k+1}), or after C^{n-1} is found.

ii) Instead of forming C, C^2, C^3, $\cdots$, C^{n-1}, form C, C^2, C^4, C^8, $\cdots$, $C^{(2^l)}$, with $l = \lceil \frac{\log(n-1)}{\log 2} \rceil$.

iii) Since only the nonzero pattern of A is of interest, to avoid unlimited increase/decrease in the elements of the successive powers, use 1's in place of the nonzero elements (Varga (2008)).

EXAMPLE

$$A = \begin{bmatrix} 0.8 & 0 & 0 & 0 & -1.2 & 0 \\ 0 & 0.9 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0.5 & 0 & -0.1 \\ 0.3 & 0 & -1.0 & 0 & 0.6 & 0 \\ 0 & 1.1 & 0 & 0 & -0.7 & 1.0 \end{bmatrix}. \quad (6)$$

$$C(A) = \text{spones}(I + \text{spones}(A)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}. \quad (7)$$

$$l \;=\; \lceil \frac{\log(n-1)}{\log 2} \rceil \;=\; \lceil 2.\cdots \rceil \;=\; 3.$$

$$\text{spones}(C^8(A)) \;=\; \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \tag{8}$$

$$\textbf{nnz}(C^8(A)) \;=\; 25 \;<\; n^2 = 36$$

A IS REDUCIBLE!

If A is **irreducible** GO TO PART 3

If A is **reducible** GO TO PART 2

PART 2: Block diagonal of **Fnf**

To form the **block diagonal** of the **Frobenius normal form** use the ideas of **(extended) compact profile technique**:
D. R. Kincaid/J. R. Respess/D. M. Young/R. G. Grimes (ACM Trans. Math. Software (1982))
A.H. (LAA (2004))

a) **perm** vector of size n. Initially it contains 1 to n in natural order. On exit it will contain a permutation of them.
b) **sizeblo** vector of (eventually) size p. On exit will contain in the position **sizeblo**(i), $i = 1(1)p$, the size of the ith diagonal block in the **Frobenius normal form**.

On exit the vectors **perm** and **sizeblo** appear as follows:

i	1	2	3	4	5	6
perm(i)	1	5	3	4	2	6

From **perm** the permutation (1, 5, 3, 4, 2, 6) gives the permutation matrix P that can produce **bdFnf**(A).

i	1	2	3
sizeblo(i)	2	1	3

$$\textbf{bdFnf}(A) = PAP^T = \left[\begin{array}{rr|r|rrr} 0.8 & -1.2 & 0 & 0 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0.7 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.5 & 0 & -0.1 \\ 0 & 0 & 0 & 0.6 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 1.1 & 1.0 \end{array}\right]. \quad (9)$$

PART 3: Identifying Classes of General $H-$ and non$-H-$Matrices

Theorem 4. *Having found the irreducible/reducible character as well as the* **bdFnf** *of a block similarity permutation of an* **Fnf** *of the given $A \in \mathbb{C}^{n,n}$, let $p =$nublo and p_1, p_2 be the number of the 1×1 blocks and the number of irreducible blocks or order ≥ 2 of* **bdFnf**, *respectively ($p = p_1 + p_2$). Suppose that all the 1×1 blocks are put first. Then, to identify the class of $H-$ or ${}_NH-$matrix to which A belongs the following steps should be performed: i) If $p_2 = 0$,* **bdFnf** *is diagonal. So, if all $a_{ii} \neq 0$ then $A \in \mathcal{H}_I$ (see 1st column of Table 3) else, that is if there is at least one of the $a_{ii} = 0$, then $A \in \mathcal{H}_S$ (see 2nd column of Table 3). END. Otherwise, if $p_2 > 0$, set a flag saying singular$= 0$. ii) If any of the diagonal elements of the last p_2 blocks of* **bdFnf** *is zero, then $A \in {}_N\mathcal{H}^0$ (see 4th column of Table 3). END. iii) If the previous case does not apply and if any of the first p_1 (1×1) blocks of* **bdFnf** *is zero, set singular$= 1$. iv) Apply an obvious slight modification of* **AH** *to each of the last p_2 blocks* *of* **bdFnf**. *a) If a block for which* $s_i < \rho(B_{ii}), i = p_1 + 1(1)p$, *is encountered then* $A \in {}_N\mathcal{H}$. *(However, if singular*$= 0$ *then* $A \in {}_N\mathcal{H}^{\emptyset}$ *and it is the case described in the 6th column of Table 3 else (singular*$= 1$*)* $A \in {}_N\mathcal{H}^0$ *it is the case described in the 5th column of Table 3.) END. b) If all the last* p_2 *blocks of* **bdFnf** *have* $s_i > \rho(B_{ii}),\ i = p_1 + 1(1)p$, *then* $A \in \mathcal{H}_I$, *provided singular*$= 0$ *else (if singular*$= 1$*)* $A \in \mathcal{H}_S$. *END. c) If there is at least one out of the last* p_2 *blocks with* $s_i = \rho(B_{ii}), i = p_1 + 1(1)p$, *and for all other blocks there hold* $s_i > \rho(B_{ii})$ *then* $A \in \mathcal{H}_M$ *(see 3rd column of Table 3) provided singular*$= 0$ *else* $A \in \mathcal{H}_S$. *END.*

$A \in \mathbb{C}^{n,n}$, $\mathcal{M}(\mathcal{F}(A)) = sI - B$, $s = \max_{i=1(1)n} \lvert a_{ii} \rvert$, $B \geq 0$					
H−matrices ($\mathcal{H}$)			*non−H−matrices* ($_N\mathcal{H}$)		
$s > \rho(B)$	$s = \rho(B)$		$s < \rho(B)$		
$\exists$ J_A[1] ($A \in \mathcal{H}_I$)	$\nexists$ J_A ($A \in \mathcal{H}_S$)	$\exists$ J_A ($A \in \mathcal{H}_M$)	$\nexists$ J_A ($A \in {_N\mathcal{H}}^{0^2}$)		$\exists$ J_A ($A \in {_N\mathcal{H}}$
A *"invertible"*	A *"singular"*	A *"mixed"*			
irreducible or reducible	− − − *reducible*	*irreducible or reducible*	*irreducible or reducible*	− − − *reducible*	*irreducible or reducible*
$a_{ii} \neq 0$ *for all* $i = 1(1)n$ *and all irreducible diagonal blocks* [4] *of* **Fnf** [5] *have* $s_i > \rho(B_{ii})$[6]	$\exists$ *at least one* 1×1 *null diagonal block* ($a_{ii} = 0$) *and all irreducible diagonal blocks of* **Fnf** *have* $s_i \geq \rho(B_{ii})$	$a_{ii} \neq 0, i = 1(1)n,$ *and all irreducible diagonal blocks of* **Fnf** *satisfy* $s_i \geq \rho(B_{ii})$ *with equality holding for at least one index* i	$\exists$ *at least one irreducible diagonal block of* **Fnf** *having at least one of its diagonal elements* $a_{ii} = 0$	$\exists$ *at least one* 1×1 *null diagonal block* ($a_{ii} = 0$) *and all irreducible diagonal blocks of* **Fnf**, *with* $s_j < \rho(B_{jj})$, ($\exists$ *at least one*) *have all their diagonal elements* $a_{kk} \neq 0$	$a_{ii} \neq 0$ *for a* $i = 1(1)n$ *an* $\exists$ *at least on irreducible diagonal bloc of* **Fnf** *with* $s_j < \rho(B_{jj})$

Numerical Examples

$$\mathcal{F}(A) = \left[\begin{array}{rrr|r|rr|r|rr|r}
3+4i & -1 & 2 & 0.1 & 0.4 & 0.7 & 1 & 1.3 & 1.6 & 1.9 \\
0 & -3 & 0.5 & 0.1 & -0.3 & -0.4 & -0.5i & 1+2i & -2 & 0.4 \\
4 & 3 & 6 & 1.1 & -1.2 & 1.5 & 2.7 & 3.8 & 0.7i & -1+2i \\ \hline
0 & 0 & 0 & i & 1+i & 1-i & 2 & 0.5 & 0.8 & -1 \\ \hline
0 & 0 & 0 & 0 & \textcircled{x} & 2 & 0.7 & 0.9 & 1.1 & 1.3i \\
0 & 0 & 0 & 0 & 3 & 8 & 2 & -1 & 3i & 7.5 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 1+i & 8 & 2.5 & -1.5 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -4.5 & 6 & 2-i \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{y}
\end{array}\right].$$

Consider the permutation

$$\begin{pmatrix} 6 & 7 & 3 & 10 & 9 & 2 & 4 & 1 & 8 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix},$$

that defines a permutation matrix P. Let $A = P\mathcal{F}(A)P^T$

$$A = \begin{bmatrix} 8 & 2 & 0 & 7.5 & 3i & 0 & 0 & 0 & -1 & 3 \\ 0 & 1+i & 0 & -1.5 & 2.5 & 0 & 0 & 0 & 8 & 0 \\ 1.5 & 2.7 & 6 & -1+2i & 0.7i & 3 & 1.1 & 4 & 3.8 & -1.2 \\ 0 & 0 & 0 & \textcircled{y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-i & 6 & 0 & 0 & 0 & -4.5 & 0 \\ -0.4 & -0.5i & 0.5 & 0.4 & -2 & -3 & 0.1 & 0 & 1+2i & -0.3 \\ 1-i & 2 & 0 & -1 & 0.8 & 0 & i & 0 & 0.5 & 1+i \\ 0.7 & 1 & 2 & 1.9 & 1.6 & -1 & 0.1 & 3+4i & 1.3 & 0.4 \\ 0 & 0 & 0 & i & 3 & 0 & 0 & 0 & 3 & 0 \\ 2 & 0.7 & 0 & 1.3i & 1.1 & 0 & 0 & 0 & 0.9 & \textcircled{x} \end{bmatrix}.$$

Applying **HornotH**, **ABGH**, **AHmod**

(x, y)	$(-1, -0.1)$	$(2, 0)$	$(-0.75, -0.1)$	$(0, 1)$	$(-0.25, 0)$	$(0.5, 1)$
$A \in$	$\mathcal{H}_I$	$\mathcal{H}_S$	$\mathcal{H}_M$	${}_N\mathcal{H}^0$	${}_N\mathcal{H}^0$	${}_N\mathcal{H}^{\emptyset}$

HAPPY $\underline{\underline{1^2 + 3^4}}$ BIRTHDAY, HANS!!!

THANK YOU VERY MUCH!

ΣΑΣ ΕΥΧΑΡΙΣΤΩ ΠΑΡΑ ΠΟΛΥ!

You ARE invited to participate in the

NUMAN 2010 CONFERENCE

(http://numan2010.science.tuc.gr)

at CHANIA on the Island of CRETE,
GREECE

September 15-18, 2010