On maximization of some functions of eigenvalues of nonnegative definite matrices

K. Filipiak[1] A. Markiewicz[1] R. Różański[2]

[1]Poznań University of Life Sciences, Poland
[2]The Higher Vocational State School in Gorzów Wlkp., Poland

26.05.2010

Notation

$$\mathbf{C} \in \mathscr{M}_v^{\geq}$$

$$\mathbf{C} = b\mathbf{I}_v - \frac{1}{b}\mathbf{S}^T\mathbf{S}, \qquad b \in \mathbb{N}$$

Notation

$$\mathbf{C} \in \mathscr{M}_v^{\geq}$$

$$\mathbf{C} = b\mathbf{I}_v - \frac{1}{b}\mathbf{S}^T\mathbf{S}, \qquad b \in \mathbb{N}$$

$\mathbf{S} = (s_{ij})_{1 \leq i,j \leq v}$

Notation

$$\mathbf{C} \in \mathscr{M}_v^{\geq}$$

$$\mathbf{C} = b\mathbf{I}_v - \frac{1}{b}\mathbf{S}^T\mathbf{S}, \qquad b \in \mathbb{N}$$

$\mathbf{S} = (s_{ij})_{1 \leq i,j \leq v}$

s_{ij} - nonnegative integer, $\qquad s_{ii} = 0$

Notation

$$\mathbf{C} \in \mathscr{M}_v^{\geq}$$

$$\mathbf{C} = b\mathbf{I}_v - \frac{1}{b}\mathbf{S}^T\mathbf{S}, \qquad b \in \mathbb{N}$$

$\mathbf{S} = (s_{ij})_{1 \leq i,j \leq v}$

s_{ij} - nonnegative integer, $\qquad s_{ii} = 0$

$\mathbf{S}\mathbf{1}_v = \mathbf{S}^T\mathbf{1}_v = b\mathbf{1}_v$

Problems

$$0 = \lambda_0(\mathbf{C}) \leq \lambda_1(\mathbf{C}) \leq \cdots \leq \lambda_{v-1}(\mathbf{C})$$

E-optimality

MAXIMIZATION OF $\lambda_1(\mathbf{C})$

Problems

$$0 = \lambda_0(\mathbf{C}) \le \lambda_1(\mathbf{C}) \le \cdots \le \lambda_{v-1}(\mathbf{C})$$

E-optimality

MAXIMIZATION OF $\lambda_1(\mathbf{C})$

D-optimality

MAXIMIZATION OF $\prod_{i=1}^{v-1} \lambda_i(\mathbf{C})$

Special classes for $b = v - 2$ and $b = v$

$\overline{\mathscr{P}}_v$ - the class of derangement matrices of order v

$$\boxed{b = v - 2}$$

$$\widetilde{\mathscr{B}} = \{\mathbf{P} : \;\; \mathbf{S} = \mathbf{1}_v \mathbf{1}_v^T - \mathbf{I}_v - \mathbf{P}, \;\; \mathbf{P} \in \overline{\mathscr{P}}_v\}$$

Special classes for $b = v - 2$ and $b = v$

$\overline{\mathscr{P}}_v$ - the class of derangement matrices of order v

$b = v - 2$

$$\widetilde{\mathscr{B}} = \{\mathbf{P} : \ \mathbf{S} = \mathbf{1}_v\mathbf{1}_v^T - \mathbf{I}_v - \mathbf{P}, \ \mathbf{P} \in \overline{\mathscr{P}}_v\}$$

$b = v$

$$\widehat{\mathscr{B}} = \{\mathbf{P} : \ \mathbf{S} = \mathbf{1}_v\mathbf{1}_v^T - \mathbf{I}_v + \mathbf{P}, \ \mathbf{P} \in \overline{\mathscr{P}}_v\}$$

Special classes

$$\mathbf{C} = b\mathbf{I}_v - \frac{1}{b}\mathbf{S}^T\mathbf{S}$$

Special classes

$$\mathbf{C} = b\mathbf{I}_v - \frac{1}{b}\mathbf{S}^T\mathbf{S}$$

$$\boxed{b = v - 2}$$

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Special classes

$$\mathbf{C} = b\mathbf{I}_v - \frac{1}{b}\mathbf{S}^T\mathbf{S}$$

$\boxed{b = v - 2}$

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$\boxed{b = v}$

$$\mathbf{S} = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 0 \end{pmatrix}$$

Special classes

$$\mathbf{C} = b\mathbf{I}_v - \frac{1}{b}\mathbf{S}^T\mathbf{S}$$

$\boxed{b = v - 2}$

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$\boxed{b = v}$

$$\mathbf{S} = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Special classes

$\mathbf{S} \in \widetilde{\mathscr{B}}$

$$\mathbf{C} = \frac{v^2 - 4v + 2}{v - 2}\mathbf{I}_v - \frac{1}{v-2}(\mathbf{P} + \mathbf{P}^T) - \frac{v-4}{v-2}\mathbf{1}_v^T\mathbf{1}_v$$

Special classes

$\mathbf{S} \in \widetilde{\mathscr{B}}$

$$\mathbf{C} = \frac{v^2 - 4v + 2}{v - 2}\mathbf{I}_v - \frac{1}{v-2}(\mathbf{P} + \mathbf{P}^T) - \frac{v-4}{v-2}\mathbf{1}_v^T\mathbf{1}_v$$

$\mathbf{S} \in \widehat{\mathscr{B}}$

$$\mathbf{C} = \frac{v^2 - 2}{v}\mathbf{I}_v + \frac{1}{v}(\mathbf{P} + \mathbf{P}^T) - \mathbf{1}_v^T\mathbf{1}_v$$

Special classes

$\mathbf{S} \in \widetilde{\mathscr{B}}$

$$\mathbf{C} = \frac{v^2 - 4v + 2}{v-2}\mathbf{I}_v - \frac{1}{v-2}(\mathbf{P} + \mathbf{P}^T) - \frac{v-4}{v-2}\mathbf{1}_v^T\mathbf{1}_v$$

$\mathbf{S} \in \widehat{\mathscr{B}}$

$$\mathbf{C} = \frac{v^2 - 2}{v}\mathbf{I}_v + \frac{1}{v}(\mathbf{P} + \mathbf{P}^T) - \mathbf{1}_v^T\mathbf{1}_v$$

$\mathbf{1}_v$ – the eigenvector associated with 0 eigenvalue.

Special classes

$\mathbf{S} \in \widetilde{\mathscr{B}}$

$$\mathbf{C} = \frac{v^2 - 4v + 2}{v-2}\mathbf{I}_v - \frac{1}{v-2}(\mathbf{P} + \mathbf{P}^T) - \frac{v-4}{v-2}\mathbf{1}_v^T\mathbf{1}_v$$

$\mathbf{S} \in \widehat{\mathscr{B}}$

$$\mathbf{C} = \frac{v^2 - 2}{v}\mathbf{I}_v + \frac{1}{v}(\mathbf{P} + \mathbf{P}^T) - \mathbf{1}_v^T\mathbf{1}_v$$

$\mathbf{1}_v$ – the eigenvector associated with 0 eigenvalue.

Special classes

$\mathbf{S} \in \widetilde{\mathscr{B}}$

$$\overline{\mathbf{C}} = \frac{v^2 - 4v + 2}{v - 2}\mathbf{I}_v - \frac{1}{v-2}(\mathbf{P} + \mathbf{P}^T)$$

$\mathbf{S} \in \widehat{\mathscr{B}}$

$$\overline{\mathbf{C}} = \frac{v^2 - 2}{v}\mathbf{I}_v + \frac{1}{v}(\mathbf{P} + \mathbf{P}^T)$$

Special classes

$\mathbf{S} \in \widetilde{\mathscr{B}}$

$$\overline{\mathbf{C}} = \frac{v^2 - 4v + 2}{v - 2}\mathbf{I}_v - \frac{1}{v-2}(\mathbf{P} + \mathbf{P}^T)$$

$\mathbf{S} \in \widehat{\mathscr{B}}$

$$\overline{\mathbf{C}} = \frac{v^2 - 2}{v}\mathbf{I}_v + \frac{1}{v}(\mathbf{P} + \mathbf{P}^T)$$

E-optimality: MAX $\lambda_1(\overline{\mathbf{C}})$

Special classes

$\mathbf{S} \in \widetilde{\mathscr{B}}$

$$\overline{\mathbf{C}} = \frac{v^2 - 4v + 2}{v - 2}\mathbf{I}_v - \frac{1}{v-2}(\mathbf{P} + \mathbf{P}^T)$$

$\mathbf{S} \in \widehat{\mathscr{B}}$

$$\overline{\mathbf{C}} = \frac{v^2 - 2}{v}\mathbf{I}_v + \frac{1}{v}(\mathbf{P} + \mathbf{P}^T)$$

E-optimality: MAX $\lambda_1(\overline{\mathbf{C}})$

D-optimality: MAX $\det \overline{\mathbf{C}}$

Two cases

$\widetilde{\mathscr{B}}$:

$$\mathbf{W} = (v-2)\overline{\mathbf{C}} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$$

$\widehat{\mathscr{B}}$: $\alpha > 2$

$$\mathbf{V} = v\overline{\mathbf{C}} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$$

Two cases

$\widetilde{\mathscr{B}}$:

$$\mathbf{W} = (v-2)\overline{\mathbf{C}} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$$

$\widehat{\mathscr{B}}$: $\alpha > 2$

$$\mathbf{V} = v\overline{\mathbf{C}} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$$

E-optimality: MAX $\lambda_1(\mathbf{W})$ (MAX $\lambda_1(\mathbf{V})$)

Two cases

$\widetilde{\mathscr{B}}$:

$$\mathbf{W} = (v-2)\overline{\mathbf{C}} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$$

$\widehat{\mathscr{B}}$: $\alpha > 2$

$$\mathbf{V} = v\overline{\mathbf{C}} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$$

E-optimality: MAX $\lambda_1(\mathbf{W})$ (MAX $\lambda_1(\mathbf{V})$)

D-optimality: MAX $\det \mathbf{W}$ (MAX $\det \mathbf{V}$)

Permutation matrices

$$\mathbf{H}_v = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

Permutation matrices

$$\mathbf{H}_v = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

$\mathbf{P} = \mathbf{H}_v$ – irreducible matrix

Permutation matrices

$$\mathbf{H}_v = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

$\mathbf{P} = \mathbf{H}_v$ – irreducible matrix

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2}),$

Permutation matrices

$$\mathbf{H}_v = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

$\mathbf{P} = \mathbf{H}_v$ – irreducible matrix

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2}), \quad \mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2} : \mathbf{H}_{v_3}),$

Permutation matrices

$$\mathbf{H}_v = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

$\mathbf{P} = \mathbf{H}_v$ – irreducible matrix

$\mathbf{P} = \text{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2}), \quad \mathbf{P} = \text{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2} : \mathbf{H}_{v_3}), \quad \dots$

– reducible matrices

E-optimality

$$\mathbf{W} = \begin{pmatrix} \alpha & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \alpha & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \alpha & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & \alpha \end{pmatrix}$$

E-optimality

$$
\mathbf{W} = \begin{pmatrix}
\alpha & -1 & 0 & 0 & 0 & 0 & -1 \\
-1 & \alpha & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & \alpha & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & \alpha & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & \alpha & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & \alpha & -1 \\
-1 & 0 & 0 & 0 & 0 & -1 & \alpha
\end{pmatrix}
$$

$$
\mathbf{W} = \left(\begin{array}{cccc|ccc}
\alpha & -1 & 0 & -1 & 0 & 0 & 0 \\
-1 & \alpha & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & \alpha & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & \alpha & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \alpha & -1 & -1 \\
0 & 0 & 0 & 0 & -1 & \alpha & -1 \\
0 & 0 & 0 & 0 & -1 & -1 & \alpha
\end{array}\right)
$$

E-optimality

$$\mathbf{W} = \begin{pmatrix} \alpha & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \alpha & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \alpha & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & \alpha \end{pmatrix}$$

$$\lambda_1(\mathbf{W})$$

$$\mathbf{W} = \left(\begin{array}{cccc|ccc} \alpha & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & \alpha \end{array}\right)$$

E-optimality

$$\mathbf{W} = \begin{pmatrix} \alpha & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \alpha & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \alpha & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & \alpha \end{pmatrix} \qquad \lambda_1(\mathbf{W})$$

$$\mathbf{W} = \left(\begin{array}{cccc|ccc} \alpha & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & \alpha \end{array}\right) \qquad \lambda_1(\mathbf{W}) = \min(\lambda_1(\mathbf{W}_1), \lambda_1(\mathbf{W}_2))$$

$\widetilde{\mathscr{B}}: \quad \mathbf{W} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \qquad k = 0, 1, 2, \ldots, v-1$$

$\widetilde{\mathscr{B}}: \quad \mathbf{W} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \qquad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha - 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\widetilde{\mathscr{B}}: \quad \mathbf{W} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \qquad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha - 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$:

$$\mu_k(\mathbf{W}) = \begin{cases} \alpha - 2\cos\frac{2k\pi}{v_1}, & k = 0, 1, 2, \ldots, v_1 - 1 \\ \alpha - 2\cos\frac{2k\pi}{v_2}, & k = 0, 1, 2, \ldots, v_2 - 1 \end{cases}$$

$\widetilde{\mathscr{B}}: \quad \mathbf{W} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \qquad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha - 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$:

$$\mu_k(\mathbf{W}) = \begin{cases} \alpha - 2\cos\frac{2k\pi}{v_1}, & k = 0, 1, 2, \ldots, v_1 - 1 \\ \alpha - 2\cos\frac{2k\pi}{v_2}, & k = 0, 1, 2, \ldots, v_2 - 1 \end{cases}$$

$\lambda_1(\mathbf{W}) = \mu_1(\mathbf{W})$ – if $\mathbf{P} = \mathbf{H}_v$ (irreducible)

$\widetilde{\mathscr{B}}: \quad \mathbf{W} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \qquad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha - 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \operatorname{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$:

$$\mu_k(\mathbf{W}) = \begin{cases} \alpha - 2\cos\frac{2k\pi}{v_1}, & k = 0, 1, 2, \ldots, v_1 - 1 \\ \alpha - 2\cos\frac{2k\pi}{v_2}, & k = 0, 1, 2, \ldots, v_2 - 1 \end{cases}$$

$\lambda_1(\mathbf{W}) = \mu_1(\mathbf{W})$ – if $\mathbf{P} = \mathbf{H}_v$ (irreducible)

$\lambda_1(\mathbf{W}) = \mu_0(\mathbf{W})$ – if $\mathbf{P}$ is reducible

$\widetilde{\mathscr{B}}: \quad \mathbf{W} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \qquad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha - 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$:

$$\mu_k(\mathbf{W}) = \begin{cases} \alpha - 2\cos\frac{2k\pi}{v_1}, & k = 0, 1, 2, \ldots, v_1 - 1 \\ \alpha - 2\cos\frac{2k\pi}{v_2}, & k = 0, 1, 2, \ldots, v_2 - 1 \end{cases}$$

$\lambda_1(\mathbf{W}) = \mu_1(\mathbf{W})$ – if $\mathbf{P} = \mathbf{H}_v$ (irreducible)

$\lambda_1(\mathbf{W}) = \mu_0(\mathbf{W})$ – if $\mathbf{P}$ is reducible

E-optimality over $\widetilde{\mathscr{B}}$

Theorem 1

Let $\overline{\mathscr{P}}_v$ be the set of permutation, derangement matrices of order v. The matrix $\mathbf{P} \in \overline{\mathscr{P}}_v$ that maximizes $\lambda_1(\mathbf{W})$ over all matrices of the form $\mathbf{W} = \alpha\mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$, $\alpha > 2$, is permutationally similar to the matrix $\mathbf{H}_v$.

E-optimality

$$\mathbf{V} = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & \alpha \end{pmatrix}$$

E-optimality

$$\mathbf{V} = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & \alpha \end{pmatrix}$$

$$\mathbf{V} = \left(\begin{array}{cccc|ccc} \alpha & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & \alpha \end{array}\right)$$

E-optimality

$$\mathbf{V} = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & \alpha \end{pmatrix}$$

$$\lambda_1(\mathbf{V})$$

$$\mathbf{V} = \left(\begin{array}{cccc|ccc} \alpha & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & \alpha \end{array}\right)$$

E-optimality

$$\mathbf{V} = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & \alpha \end{pmatrix} \qquad \lambda_1(\mathbf{V})$$

$$\mathbf{V} = \left(\begin{array}{cccc|ccc} \alpha & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & \alpha \end{array}\right) \qquad \lambda_1(\mathbf{V}) = \min(\lambda_1(\mathbf{V}_1), \lambda_1(\mathbf{V}_2))$$

$\widehat{\mathscr{B}}: \quad \mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\widehat{\mathscr{B}}: \quad \mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha + 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\widehat{\mathscr{B}}: \quad \mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha + 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$:

$$\mu_k(\mathbf{W}) = \begin{cases} \alpha + 2\cos\frac{2k\pi}{v_1}, & k = 0, 1, 2, \ldots, v_1 - 1 \\ \alpha + 2\cos\frac{2k\pi}{v_2}, & k = 0, 1, 2, \ldots, v_2 - 1 \end{cases}$$

$\widehat{\mathscr{B}}$: $\mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha + 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$:

$$\mu_k(\mathbf{W}) = \begin{cases} \alpha + 2\cos\frac{2k\pi}{v_1}, & k = 0, 1, 2, \ldots, v_1 - 1 \\ \alpha + 2\cos\frac{2k\pi}{v_2}, & k = 0, 1, 2, \ldots, v_2 - 1 \end{cases}$$

$\lambda_1(\mathbf{V}) = \mu_{v/2}(\mathbf{V})$ – if v is even

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha + 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$:

$$\mu_k(\mathbf{W}) = \begin{cases} \alpha + 2\cos\frac{2k\pi}{v_1}, & k = 0, 1, 2, \ldots, v_1 - 1 \\ \alpha + 2\cos\frac{2k\pi}{v_2}, & k = 0, 1, 2, \ldots, v_2 - 1 \end{cases}$$

$\lambda_1(\mathbf{V}) = \mu_{v/2}(\mathbf{V})$ – if v is even

$\lambda_1(\mathbf{V}) = \mu_{(v\pm1)/2}(\mathbf{V})$ – if v is odd

$\widehat{\mathscr{B}}: \quad \mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

$$\mu_k(\mathbf{H}_v + \mathbf{H}_v^T) = 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathbf{H}_v$:

$$\mu_k(\mathbf{W}) = \alpha + 2\cos\frac{2k\pi}{v}, \quad k = 0, 1, 2, \ldots, v-1$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$:

$$\mu_k(\mathbf{W}) = \begin{cases} \alpha + 2\cos\frac{2k\pi}{v_1}, & k = 0, 1, 2, \ldots, v_1 - 1 \\ \alpha + 2\cos\frac{2k\pi}{v_2}, & k = 0, 1, 2, \ldots, v_2 - 1 \end{cases}$$

$\lambda_1(\mathbf{V}) = \mu_{v/2}(\mathbf{V})$ – if v is even

$\lambda_1(\mathbf{V}) = \mu_{(v\pm1)/2}(\mathbf{V})$ – if v is odd

E-optimality over $\widehat{\mathscr{B}}$

Theorem 2

Let $\overline{\mathscr{P}_v}$ be the set of permutation, derangement matrices of order v. The matrix $\mathbf{P} \in \overline{\mathscr{P}_v}$ that maximizes $\lambda_1(\mathbf{V})$ over all matrices of the form $\mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$, $\alpha > 2$, is permutationally similar to the matrix

(i) $\mathbf{H}_v$ if $v = 2, 7$;

(ii) $\mathbf{I}_2 \otimes \mathbf{H}_2$ or $\mathbf{H}_4$ if $v = 4$;

(iii) $\mathbf{I}_m \otimes \mathbf{H}_3$ if $v = 3m$, $m \in \mathbb{N}$;

(iv) $\mathrm{diag}(\mathbf{I}_i \otimes \mathbf{H}_3, \mathbf{I}_j \otimes \mathbf{H}_5)$ if $v = 5$ or $v \geq 8$ and $v \neq 3m$, $m \in \mathbb{N}$, with $v = 3i + 5j$ for some $i \in \mathbb{N} \cup \{0\}$, $j \in \mathbb{N}$.

D-optimality

$$\mathbf{W} = \begin{pmatrix} \alpha & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \alpha & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \alpha & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & \alpha \end{pmatrix}$$

D-optimality

$$\mathbf{W} = \begin{pmatrix} \alpha & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \alpha & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \alpha & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & \alpha \end{pmatrix}$$

$$\mathbf{W} = \left(\begin{array}{cccc|ccc} \alpha & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & \alpha \end{array}\right)$$

D-optimality

$$\mathbf{W} = \begin{pmatrix} \alpha & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \alpha & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \alpha & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & \alpha \end{pmatrix}$$

$$\det \mathbf{W}$$

$$\mathbf{W} = \left(\begin{array}{cccc|ccc} \alpha & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & \alpha \end{array}\right)$$

D-optimality

$$\mathbf{W} = \begin{pmatrix} \alpha & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \alpha & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \alpha & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & \alpha \end{pmatrix}$$

$$\det \mathbf{W}$$

$$\mathbf{W} = \left(\begin{array}{cccc|ccc} \alpha & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & \alpha & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \alpha & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & \alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & \alpha & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & \alpha \end{array}\right)$$

$$\det \mathbf{W} = \det \mathbf{W}_1 \cdot \det \mathbf{W}_2$$

$\mathbf{W} = \alpha\mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$

Determinant of tridiagonal matrix with corners (Molinari, 2008):

$$\det\left(\alpha\mathbf{I}_v - (\mathbf{H}_v + \mathbf{H}_v^T)\right) = -2 + \mathrm{tr}\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}^v\right]$$

$\mathbf{W} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$

Determinant of tridiagonal matrix with corners (Molinari, 2008):

$$\det\left(\alpha \mathbf{I}_v - (\mathbf{H}_v + \mathbf{H}_v^T)\right) = -2 + \mathrm{tr}\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}^v\right]$$
$$= -2 + x^v + y^v$$

$$x = \lambda_0\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}\right] = \frac{\alpha - \sqrt{\alpha^2 - 4}}{2}, \qquad x \in (0, \tfrac{1}{2})$$
$$y = \lambda_1\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}\right] = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2} = \frac{1}{x}, \qquad y \in (\alpha - \tfrac{1}{2}, \alpha)$$

$\mathbf{W} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$

Determinant of tridiagonal matrix with corners (Molinari, 2008):

$$\det\left(\alpha \mathbf{I}_v - (\mathbf{H}_v + \mathbf{H}_v^T)\right) = -2 + \text{tr}\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}^v\right]$$

$$= -2 + x^v + y^v$$

$$x = \lambda_0\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}\right] = \frac{\alpha - \sqrt{\alpha^2 - 4}}{2}, \qquad x \in (0, \tfrac{1}{2})$$

$$y = \lambda_1\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}\right] = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2} = \frac{1}{x}, \qquad y \in (\alpha - \tfrac{1}{2}, \alpha)$$

$$-2 + x^v + y^v \geq (-2 + x^{v_1} + y^{v_1})(-2 + x^{v_2} + y^{v_2})$$

D-optimality over $\widetilde{\mathscr{B}}$

Theorem 3

Let $\overline{\mathscr{P}}_v$ be the set of permutation, derangement matrices of order v. The matrix $\mathbf{P} \in \overline{\mathscr{P}}_v$ that maximizes $\det \mathbf{W}$ over all matrices of the form $\mathbf{W} = \alpha \mathbf{I}_v - (\mathbf{P} + \mathbf{P}^T)$, $\alpha > 2$, is permutationally similar to the matrix $\mathbf{H}_v$.

$\mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

Determinant of tridiagonal matrix with corners (Molinari, 2008):

$$\det\left(\alpha \mathbf{I}_v + (\mathbf{H}_v + \mathbf{H}_v^T)\right) = 2 \cdot (-1)^{v+1} + \mathrm{tr}\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}^v\right]$$

$\mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

Determinant of tridiagonal matrix with corners (Molinari, 2008):

$$\det\left(\alpha \mathbf{I}_v + (\mathbf{H}_v + \mathbf{H}_v^T)\right) = 2 \cdot (-1)^{v+1} + \mathrm{tr}\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}^v\right]$$

$$= 2 \cdot (-1)^{v+1} + x^v + y^v$$

$$x = \lambda_0\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}\right] = \frac{\alpha - \sqrt{\alpha^2 - 4}}{2}, \qquad x \in (0, \tfrac{1}{2})$$

$$y = \lambda_1\left[\begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix}\right] = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2} = \frac{1}{x}, \qquad y \in (\alpha - \tfrac{1}{2}, \alpha)$$

$\mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

$$\det\left(\alpha \mathbf{I}_v + (\mathbf{H}_v + \mathbf{H}_v^T)\right) = \begin{cases} -2 + x^v + y^v, & v \text{ even} \\ 2 + x^v + y^v, & v \text{ odd} \end{cases}$$

$\mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

$$\det\left(\alpha \mathbf{I}_v + (\mathbf{H}_v + \mathbf{H}_v^T)\right) = \begin{cases} -2 + x^v + y^v, & v \text{ even} \\ 2 + x^v + y^v, & v \text{ odd} \end{cases}$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$, v_i – odd:

$$-2 + x^v + y^v \leq (2 + x^{v_1} + y^{v_1})(2 + x^{v_2} + y^{v_2})$$

$\mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$

$$\det\left(\alpha \mathbf{I}_v + (\mathbf{H}_v + \mathbf{H}_v^T)\right) = \begin{cases} -2 + x^v + y^v, & v \text{ even} \\ 2 + x^v + y^v, & v \text{ odd} \end{cases}$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$, v_i – odd:

$$-2 + x^v + y^v \leq (2 + x^{v_1} + y^{v_1})(2 + x^{v_2} + y^{v_2})$$

$\mathbf{P} = \mathrm{diag}(\mathbf{H}_{v_1} : \mathbf{H}_{v_2})$, v_1 – odd, v_2 – even:

$$2 + x^v + y^v \leq (2 + x^{v_1} + y^{v_1})(-2 + x^{v_2} + y^{v_2})$$

D-optimality over $\widehat{\mathscr{B}}$

Theorem 4

Let $\overline{\mathscr{P}}_v$ be the set of permutation, derangement matrices of order v. The matrix $\mathbf{P} \in \overline{\mathscr{P}}_v$ that maximizes $\det \mathbf{V}$ over all matrices of the form $\mathbf{V} = \alpha \mathbf{I}_v + (\mathbf{P} + \mathbf{P}^T)$, $\alpha > 2$, is permutationally similar to the matrix

(i) $\mathbf{I}_m \otimes \mathbf{H}_3$ if $v = 3m$, $m \in \mathbb{N}$;

(ii) $\mathrm{diag}(\mathbf{I}_m \otimes \mathbf{H}_3, \mathbf{H}_4)$ if $v = 3m + 4$, $m \in \mathbb{N} \cup \{0\}$;

(iii) $\mathrm{diag}(\mathbf{I}_m \otimes \mathbf{H}_3, \mathbf{H}_5)$ if $v = 3m + 5$, $m \in \mathbb{N} \cup \{0\}$.

Experiment

- v - number of treatments

Experiment

- v - number of treatments
- bk - number of units

Experiment

- v - number of treatments
- bk - number of units
- b - number of blocks

Experiment

- v - number of treatments
- bk - number of units
- b - number of blocks
- k - number of units in each block (block size)

Experiment

- v - number of treatments
- bk - number of units
- b - number of blocks
- k - number of units in each block (block size)

- $\mathscr{D}_{v,b,k}$ - the class of block designs

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

- $\mathbf{y}$ - vector of observations

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

- $\mathbf{y}$ - vector of observations
- τ - vector of treatment effects

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

- $\mathbf{y}$ - vector of observations
- τ - vector of treatment effects
- λ - vector of left neighbor effects

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

- $\mathbf{y}$ - vector of observations
- τ - vector of treatment effects
- λ - vector of left neighbor effects
- β - vector of block effects

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

- $\mathbf{y}$ - vector of observations
- τ - vector of treatment effects
- λ - vector of left neighbor effects
- β - vector of block effects
- ε - vector of random errors, $\varepsilon \sim N\left(\mathbf{0}, \sigma^2\mathbf{I}_{bk}\right)$

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

- $\mathbf{y}$ - vector of observations
- τ - vector of treatment effects
- λ - vector of left neighbor effects
- β - vector of block effects
- ε - vector of random errors, $\varepsilon \sim N\left(\mathbf{0}, \sigma^2\mathbf{I}_{bk}\right)$
- $\mathbf{T}_d$ - the design matrix of treatment effects

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

- $\mathbf{y}$ - vector of observations
- τ - vector of treatment effects
- λ - vector of left neighbor effects
- β - vector of block effects
- ε - vector of random errors, $\varepsilon \sim N\left(\mathbf{0}, \sigma^2\mathbf{I}_{bk}\right)$
- $\mathbf{T}_d$ - the design matrix of treatment effects
- $\mathbf{L}_d$ - the design matrix of left neighbor effects

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

- $\mathbf{y}$ - vector of observations
- τ - vector of treatment effects
- λ - vector of left neighbor effects
- β - vector of block effects
- ε - vector of random errors, $\varepsilon \sim N\left(\mathbf{0}, \sigma^2\mathbf{I}_{bk}\right)$
- $\mathbf{T}_d$ - the design matrix of treatment effects
- $\mathbf{L}_d$ - the design matrix of left neighbor effects
- $\mathbf{B} = \mathbf{I}_b \otimes \mathbf{1}_k$ - the design matrix of block effects

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + \mathbf{B}\beta + \varepsilon$$

$$\mathbf{L}_d = (\mathbf{I}_b \otimes \mathbf{H}_k)\mathbf{T}_d$$

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d\boldsymbol{\tau} + \mathbf{L}_d\boldsymbol{\lambda} + \mathbf{B}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$$\mathbf{L}_d = (\mathbf{I}_b \otimes \mathbf{H}_k)\mathbf{T}_d$$

$$\mathbf{H}_k = \begin{pmatrix} \mathbf{0}'_{k-1} & 1 \\ \mathbf{I}_{k-1} & \mathbf{0}_{k-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Interference model

Interference model

$$\mathbf{y} = \mathbf{T}_d \boldsymbol{\tau} + \mathbf{L}_d \boldsymbol{\lambda} + \mathbf{B}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$$\mathbf{L}_d = (\mathbf{I}_b \otimes \mathbf{H}_k)\mathbf{T}_d$$

$$\mathbf{H}_k = \begin{pmatrix} \mathbf{0}'_{k-1} & 1 \\ \mathbf{I}_{k-1} & \mathbf{0}_{k-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$\mathbf{H}_k$ - the circular incidence matrix

Information matrix for the estimation of treatment effects

$$\mathbf{U}_d = \left(\begin{array}{c|cc} \mathbf{T}_d^T\mathbf{T}_d & \mathbf{T}_d^T\mathbf{L}_d & \mathbf{T}_d^T\mathbf{B} \\ \hline \mathbf{L}_d^T\mathbf{T}_d & \mathbf{L}_d^T\mathbf{L}_d & \mathbf{L}_d^T\mathbf{B} \\ \mathbf{B}^T\mathbf{T}_d & \mathbf{B}^T\mathbf{L}_d & \mathbf{B}^T\mathbf{B} \end{array}\right)$$

Information matrix for the estimation of treatment effects

$$\mathbf{U}_d = \left(\begin{array}{c|cc} \mathbf{T}_d^T\mathbf{T}_d & \mathbf{T}_d^T\mathbf{L}_d & \mathbf{T}_d^T\mathbf{B} \\ \hline \mathbf{L}_d^T\mathbf{T}_d & \mathbf{L}_d^T\mathbf{L}_d & \mathbf{L}_d^T\mathbf{B} \\ \mathbf{B}^T\mathbf{T}_d & \mathbf{B}^T\mathbf{L}_d & \mathbf{B}^T\mathbf{B} \end{array}\right) = \left(\begin{array}{c|cc} \mathbf{R}_d & \mathbf{S}_d^T & \mathbf{N}_d \\ \hline \mathbf{S}_d & \mathbf{R}_d & \mathbf{N}_d \\ \mathbf{N}_d^T & \mathbf{N}_d^T & \mathbf{K} \end{array}\right)$$

Information matrix for the estimation of treatment effects

$$\mathbf{U}_d = \left(\begin{array}{c|cc} \mathbf{T}_d^T\mathbf{T}_d & \mathbf{T}_d^T\mathbf{L}_d & \mathbf{T}_d^T\mathbf{B} \\ \hline \mathbf{L}_d^T\mathbf{T}_d & \mathbf{L}_d^T\mathbf{L}_d & \mathbf{L}_d^T\mathbf{B} \\ \mathbf{B}^T\mathbf{T}_d & \mathbf{B}^T\mathbf{L}_d & \mathbf{B}^T\mathbf{B} \end{array}\right) = \left(\begin{array}{c|cc} \mathbf{R}_d & \mathbf{S}_d^T & \mathbf{N}_d \\ \hline \mathbf{S}_d & \mathbf{R}_d & \mathbf{N}_d \\ \mathbf{N}_d^T & \mathbf{N}_d^T & \mathbf{K} \end{array}\right)$$

$$\mathbf{R}_d = \mathrm{diag}(r_1, \ldots, r_v), \qquad \mathbf{K} = k\mathbf{I}_b$$

Information matrix for the estimation of treatment effects

$$\mathbf{U}_d = \left(\begin{array}{c|cc} \mathbf{T}_d^T\mathbf{T}_d & \mathbf{T}_d^T\mathbf{L}_d & \mathbf{T}_d^T\mathbf{B} \\ \hline \mathbf{L}_d^T\mathbf{T}_d & \mathbf{L}_d^T\mathbf{L}_d & \mathbf{L}_d^T\mathbf{B} \\ \mathbf{B}^T\mathbf{T}_d & \mathbf{B}^T\mathbf{L}_d & \mathbf{B}^T\mathbf{B} \end{array} \right) = \left(\begin{array}{c|cc} \mathbf{R}_d & \mathbf{S}_d^T & \mathbf{N}_d \\ \hline \mathbf{S}_d & \mathbf{R}_d & \mathbf{N}_d \\ \mathbf{N}_d^T & \mathbf{N}_d^T & \mathbf{K} \end{array} \right)$$

$$\mathbf{R}_d = \mathrm{diag}(r_1, \ldots, r_v), \qquad \mathbf{K} = k\mathbf{I}_b$$

$$\mathbf{C}_d = \left[\mathbf{U}_d / \begin{pmatrix} \mathbf{R}_d & \mathbf{N}_d \\ \mathbf{N}_d^T & \mathbf{K} \end{pmatrix} \right] = \mathbf{R}_d - \mathbf{S}_d^T\mathbf{R}_d\mathbf{S}_d -$$

$$-(\mathbf{N}_d - \mathbf{S}_d^T\mathbf{R}_d^{-1}\mathbf{N}_d)(\mathbf{K} - \mathbf{N}_d^T\mathbf{R}_d^{-1}\mathbf{N}_d)^-(\mathbf{N}_d - \mathbf{S}_d^T\mathbf{R}_d^{-1}\mathbf{N}_d)^T$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \\ \mathrm{A} & \mathrm{C} & \mathrm{B} & \mathrm{D} \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \\ \mathrm{A} & \mathrm{C} & \mathrm{B} & \mathrm{D} \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} \text{A} & \text{B} & \text{C} & \text{D} \\ \text{A} & \text{C} & \text{B} & \text{D} \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} \text{A} & \text{B} & \text{C} & \text{D} \\ \text{A} & \text{C} & \text{B} & \text{D} \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \; b = 2$

$$d = \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \\ \mathrm{A} & \mathrm{C} & \mathrm{B} & \mathrm{D} \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & 0 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \; b = 2$

$$d = \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \\ \mathrm{A} & \mathrm{C} & \mathrm{B} & \mathrm{D} \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} \text{A} & \text{B} & \text{C} & \text{D} \\ \text{A} & \text{C} & \text{B} & \text{D} \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} {\color{red}0} & 1 & 1 & 0 \\ 0 & {\color{red}0} & 1 & 1 \\ {\color{blue}0} & & {\color{red}0} & \\ & & & {\color{red}0} \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \\ \mathrm{A} & \mathrm{C} & \mathrm{B} & \mathrm{D} \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \\ & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} \text{A} & \text{B} & \text{C} & \text{D} \\ \text{A} & \text{C} & \text{B} & \text{D} \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & & 0 \end{pmatrix}$$

Example - Complete binary design

$v = k = 4, \;\; b = 2$

$$d = \begin{pmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} \\ \mathrm{A} & \mathrm{C} & \mathrm{B} & \mathrm{D} \end{pmatrix}$$

$\mathbf{N}_d = \mathbf{T}_d^T \mathbf{B} = \mathbf{1}_v \mathbf{1}_b^T$

$$\mathbf{S}_d = \mathbf{T}_d^T \mathbf{L}_d = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

Design - general case

$$\mathbf{N}_d \mathbf{1}_b = \begin{pmatrix} r_1 \\ \vdots \\ r_v \end{pmatrix}, \qquad \mathbf{N}_d^T \mathbf{1}_v = v \mathbf{1}_b$$

Design - general case

$$\mathbf{N}_d\mathbf{1}_b = \begin{pmatrix} r_1 \\ \vdots \\ r_v \end{pmatrix}, \qquad \mathbf{N}_d^T\mathbf{1}_v = v\mathbf{1}_b$$

$$\mathbf{S}_d\mathbf{1}_v = \mathbf{S}_d^T\mathbf{1}_v = \begin{pmatrix} r_1 \\ \vdots \\ r_v \end{pmatrix}, \qquad s_{ii} \neq 0$$

Design - general case

$$\mathbf{N}_d\mathbf{1}_b = \begin{pmatrix} r_1 \\ \vdots \\ r_v \end{pmatrix}, \qquad \mathbf{N}_d^T\mathbf{1}_v = v\mathbf{1}_b$$

$$\mathbf{S}_d\mathbf{1}_v = \mathbf{S}_d^T\mathbf{1}_v = \begin{pmatrix} r_1 \\ \vdots \\ r_v \end{pmatrix}, \qquad s_{ii} \neq 0$$

$$\sum_{i=1}^{v}\sum_{j=1}^{b} n_{ij} = \sum_{i=1}^{v}\sum_{j=1}^{v} s_{ij} = bv$$

$\mathbf{C}_d$ of complete binary design

$$\mathbf{C}_d = \left[\mathbf{U}_d / \begin{pmatrix} \mathbf{R}_d & \mathbf{N}_d \\ \mathbf{N}_d^T & \mathbf{K} \end{pmatrix}\right] = \mathbf{R}_d - \mathbf{S}_d^T \mathbf{R}_d \mathbf{S}_d - $$
$$-(\mathbf{N}_d - \mathbf{S}_d^T \mathbf{R}_d^{-1} \mathbf{N}_d)(\mathbf{K} - \mathbf{N}_d^T \mathbf{R}_d^{-1} \mathbf{N}_d)^{-}(\mathbf{N}_d - \mathbf{S}_d^T \mathbf{R}_d^{-1} \mathbf{N}_d)^T$$

$\mathbf{C}_d$ of complete binary design

$$\mathbf{C}_d = \left[\mathbf{U}_d / \begin{pmatrix} \mathbf{R}_d & \mathbf{N}_d \\ \mathbf{N}_d^T & \mathbf{K} \end{pmatrix}\right] = \mathbf{R}_d - \mathbf{S}_d^T \mathbf{R}_d \mathbf{S}_d -$$

$$-(\mathbf{N}_d - \mathbf{S}_d^T \mathbf{R}_d^{-1} \mathbf{N}_d)(\mathbf{K} - \mathbf{N}_d^T \mathbf{R}_d^{-1} \mathbf{N}_d)^{-}(\mathbf{N}_d - \mathbf{S}_d^T \mathbf{R}_d^{-1} \mathbf{N}_d)^T$$

Binary design with $v = k$: $\mathbf{R}_d = b\mathbf{I}_v$

$\mathbf{C}_d$ of complete binary design

$$\mathbf{C}_d = \left[\mathbf{U}_d / \begin{pmatrix} \mathbf{R}_d & \mathbf{N}_d \\ \mathbf{N}_d^T & \mathbf{K} \end{pmatrix}\right] = \mathbf{R}_d - \mathbf{S}_d^T \mathbf{R}_d \mathbf{S}_d -$$
$$-(\mathbf{N}_d - \mathbf{S}_d^T \mathbf{R}_d^{-1} \mathbf{N}_d)(\mathbf{K} - \mathbf{N}_d^T \mathbf{R}_d^{-1} \mathbf{N}_d)^{-}(\mathbf{N}_d - \mathbf{S}_d^T \mathbf{R}_d^{-1} \mathbf{N}_d)^T$$

Binary design with $v = k$: $\mathbf{R}_d = b\mathbf{I}_v$

$$\mathbf{C}_d = b\mathbf{I}_v - b^{-1} \mathbf{S}_d^T \mathbf{S}_d$$

E, D-optimality over $\mathscr{D}_{v,v-2,v}$

Theorem

If there exists design d^* with the left-neighboring matrix $\mathbf{S}_{d^*} = \mathbf{1}_v\mathbf{1}_v^T - \mathbf{I}_v - \mathbf{P}_{d^*}$, such that $\mathbf{P}_{d^*}$ is permutationally similar to the matrix given in Theorem 1, then d^* is E-optimal over the class $\mathscr{D}_{v,v-2,v}$.

E, D-optimality over $\mathscr{D}_{v,v-2,v}$

Theorem

If there exists design d^* with the left-neighboring matrix $\mathbf{S}_{d^*} = \mathbf{1}_v\mathbf{1}_v^T - \mathbf{I}_v - \mathbf{P}_{d^*}$, such that $\mathbf{P}_{d^*}$ is permutationally similar to the matrix given in Theorem 1, then d^* is E-optimal over the class $\mathscr{D}_{v,v-2,v}$.

Theorem

If there exists design d^* with the left-neighboring matrix $\mathbf{S}_{d^*} = \mathbf{1}_v\mathbf{1}_v^T - \mathbf{I}_v + \mathbf{P}_{d^*}$, such that $\mathbf{P}_{d^*}$ is permutationally similar to the matrix given in Theorem 3, then d^* is D-optimal over the class $\mathscr{D}_{v,v-2,v}$.

E, D-optimality over $\mathscr{D}_{v,v,v}$

Theorem

If there exists design d^* with the left-neighboring matrix $\mathbf{S}_{d^*} = \mathbf{1}_v\mathbf{1}_v^T - \mathbf{I}_v - \mathbf{P}_{d^*}$, such that $\mathbf{P}_{d^*}$ is permutationally similar to the matrix given in Theorem 2, then d^* is E-optimal over the class $\mathscr{D}_{v,v,v}$.

E, D-optimality over $\mathscr{D}_{v,v,v}$

Theorem

If there exists design d^* with the left-neighboring matrix $\mathbf{S}_{d^*} = \mathbf{1}_v \mathbf{1}_v^T - \mathbf{I}_v - \mathbf{P}_{d^*}$, such that $\mathbf{P}_{d^*}$ is permutationally similar to the matrix given in Theorem 2, then d^* is E-optimal over the class $\mathscr{D}_{v,v,v}$.

Theorem

If there exists design d^* with the left-neighboring matrix $\mathbf{S}_{d^*} = \mathbf{1}_v \mathbf{1}_v^T - \mathbf{I}_v + \mathbf{P}_{d^*}$, such that $\mathbf{P}_{d^*}$ is permutationally similar to the matrix given in Theorem 4, then d^* is D-optimal over the class of all equireplicated desings.

Main References

- Filipiak, K., R. Różański, A. Sawikowska, and D. Wojtera-Tyrakowska (2008). On the E-optimality of complete designs under an interference model. *Statist. Probab. Lett.* **78**, 2470–2477.
- Molinari, L.G. (2008). Determinants of block tridiagonal matrices. *Linear Algebra Appl.* **429**, 2221–2226.