Eigenvalue localization and some classes of matrices related to positivity

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Definition. A real matrix is a **P-matrix** if all its principal minors are positive

Some classes of P-matrices:

C1: Symmetric positive definite matrices.

A matrix is *totally positive* if all its minors are nonnegative.
C2: Nonsingular totally positive matrices.

A nonsingular matrix A with positive diagonal elements and nonpositive off-diagonal elements is an M-matrix if $A^{-1} \geq 0$.
C3: Nonsingular M-matrices.

C4: Matrices with positive diagonal elements which are strictly diagonal dominant by rows.

C5: Matrices with positive row sums and all its off-diagonal elements bounded above by the corresponding row means.

Principal submatrices inherit these properties.

Localization of the real parts of the eigenvalues

Definition. We say that a square real matrix $A = (a_{ik})_{1\le i,k\le n}$ with positive row sums is a **B-matrix** if all its off-diagonal elements are bounded above by the corresponding row means, i.e., for all $i = 1, \dots, n$

$$\sum_{k=1}^{n} a_{ik} > 0, \quad \text{and} \quad \frac{1}{n}\left(\sum_{k=1}^{n} a_{ik}\right) > a_{ij} \ \forall j \neq i.$$

In [1] it was proved that a B-matrix is nonsingular and has positive determinant. The definition of an $n \times n$ B-matrix involves n^2 linear inequalities. It was also proved that this set of inequalities forms a **weakest set of linear conditions** on the rows of a real $n \times n$ matrix **to ensure positive determinant**. Another weakest set of linear conditions to ensure positive determinant is provided by the $n2^{n-1}$ inequalities corresponding to strict diagonal dominance by rows with positive diagonal entries.

[1] Linear conditions for positive determinants (J.M. Carnicer , T.N.T. Goodman, and J.M. P.), *Linear Algebra Appl.* **292** (1999), pp. 39-59.

[2] A class of P-matrices with applications to the localization of of the eigenvalues of a real matrix (J.M. P.). *SIAM J. Matrix Anal. Appl.* **22**, pp. 1027-1037.

$r_i^+ := \max\{0, a_{ij} \mid j \neq i\}$.

PROPOSITION. *Let* $A = (a_{ik})_{1 \leq i,k \leq n}$ *be a real matrix. Then* A *is a* *B-matrix if and only if, for all* $i \in \{1, \ldots, n\}$, $\sum_{k=1}^{n} a_{ik} > n r_i^+$.

EXAMPLE. $A = \begin{pmatrix} 1+\varepsilon & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1+\varepsilon \end{pmatrix}$

THEOREM. *Any principal submatrix of a B-matrix is a B-matrix.*

COROLLARY. *B-matrices are P-matrices.*

DEFINITION. We say that a real matrix is a *$\bar{B}$-matrix* if it is of the form DA where D is a diagonal matrix whose diagonal elements belong to the set $\{1, -1\}$ and A is a B-matrix.

Let $A = (a_{ik})_{1 \le i,k \le n}$ be a real matrix. For each $i = 1, \dots, n$

$$r_i^+ := \max\{0, a_{ij} \mid j \neq i\}, \; r_i^- := \min\{0, a_{ij} \mid j \neq i\},$$

$$r_i := \begin{cases} r_i^+ & \text{if } a_{ii} > 0 \\ r_i^- & \text{if } a_{ii} < 0 \end{cases} \tag{1}$$

PROPOSITION. *Let $A = (a_{ik})_{1 \le i,k \le n}$ be a real matrix and let r_i be as in (1). Then A is a $\bar{B}$-matrix if and only if for all $i = 1, \dots, n$*

$$|a_{ii} - r_i| > \sum_{j \neq i} |r_i - a_{ij}|.$$

$$r_i^+ := \max\{0, a_{ij} \mid j \neq i\},\ r_i^- := \min\{0, a_{ij} \mid j \neq i\} \tag{1}$$

Given a matrix $B = (b_{ik})_{1\leq i,k\leq n}$, let us define the family of matrices

$$B_t := D + t(B - D), \quad t \in [0, 1], (2)$$

where D is the diagonal matrix $\mathrm{diag}\{b_{11}, \ldots, b_{nn}\}$.

THEOREM. *Let $A = (a_{ik})_{1\leq i,k\leq n}$ be a real matrix, let r_i^+, r_i^- be as in (1) and let λ be a real eigenvalue of A. Then:*

(i) $\lambda \in S := \bigcup_{i=1}^{n} [a_{ii} - r_i^+ - \sum_{k\neq i} |r_i^+ - a_{ik}|\,,\ a_{ii} - r_i^- + \sum_{k\neq i} |r_i^- - a_{ik}|]$.

(ii) *Let $\mathcal{C}$ be a class of real matrices such that if $B \in \mathcal{C}$ then all eigenvalues of B are real and all matrices of the form (2) belong to $\mathcal{C}$ and let us assume that $A \in \mathcal{C}$. If S' is the union of m intervals of S such that S' is disjoint from all other intervals, then S' contains precisely m eigenvalues (counting multiplicities) of A.*

Richard S. Varga and Alan Krautstengel, On Gerschgorin-type problems and ovals of Cassini, *ETNA* **8** (1999), pp. 15–20.

J.M. P., On an alternative to Gerschgorin circles and ovals of Cassini, *Numer. Math.* **95**, pp. 337-345.

A matrix $A = (a_{ik})_{1\leq i,k\leq n}$ with diagonal elements satisfying $a_{kk} > r_k^+$ for all k is a *doubly B-matrix* if, for all $i \neq j$ in $\{1, \ldots, n\}$,

$$(a_{ii} - r_i^+)(a_{jj} - r_j^+) > \Big(\sum_{k\neq i}(r_i^+ - a_{ik})\Big)\Big(\sum_{k\neq j}(r_j^+ - a_{jk})\Big),$$

Theorem. *If A is a doubly B-matrix then* $\det A > 0$ *and A is a P-matrix*.

We have proved a result on the localization of the real eigenvalues of a real matrix of a nature similar to the ovals of Cassini derived by Brauer.

H.-B. Li, T.-Z. Huang and H. Li, *On some subclasses of P-matrices*. Numer. Linear Algebra Appl. 14 (2007), 391–405.

Other alternative extension of B-matrices

L. Cvetković and J.M. P.: "Minimal sets alternative to minimal Geršgorin sets". *Applied Numerical Mathematics* 60, 442-451.

A real square matrix is an **H_+B-matrix** if there exists a diagonal matrix X with positive diagonal entries such that AX is a B-matrix.

We discuss the possibility of finding a scaling matrix X, which scales an H_+B-matrix into a B-matrix. For this purpose, we present a subclass of H_+B-matrices, for which we are able to find the exact form of a scaling matrix. This subclass was called **B^S-matrices**.

J.M. P.: "Exclusion and inclusion intervals for the real eigenvalues of positive matrices" (2005). *SIAM Journal on Matrix Analysis and its Applications* **26**, pp. 908-917.

Definition. We say that a square real matrix $A = (a_{ik})_{1 \le i,k \le n}$ with positive row sums is a *C-matrix* if all its off-diagonal elements are greater than the corresponding row means, i.e., for all $i = 1, \dots, n$

$$0 < \frac{1}{n}\left(\sum_{k=1}^{n} a_{ik}\right) < a_{ij} \quad \forall j \neq i.$$

THEOREM. *If A is an $n \times n$ C-matrix, then* $(-1)^{n-1} \det A > 0$

EXAMPLE. $A = \begin{pmatrix} 1-\varepsilon & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1-\varepsilon \end{pmatrix}$

$$s_i^+ := \max\{0, \min\{a_{ij} \mid j \neq i\}\}v, \quad s_i^- := \min\{0, \max\{a_{ij} \mid j \neq i\}\}$$

PROPOSITION. *Let $A = (a_{ik})_{1 \le i,k \le n}$ be a real matrix. Then A is a C-matrix if and only if for all $i \in \{1, \dots, n\}$*

$$0 < \sum_{k=1}^{n} a_{ik} < n s_i^+ .$$

Definition. We say that a real matrix is a $\bar{C}$-matrix if it is of the form DA where D is a diagonal matrix whose diagonal elements belong to the set $\{1, -1\}$ and A is a C-matrix.

Since C-matrices are not P-matrices, we only localize the real eigenvalues:

THEOREM. Let $A = (a_{ik})_{1 \le i,k \le n}$ be a real matrix and λ a real eigenvalue of A. Then λ does not belong to the interval

$$(\max_i\{a_{ii} - s_i^+ - \sum_{k \neq i} |a_{ik} - s_i^+|\} \quad \min_i\{a_{ii} - s_i^- + \sum_{k \neq i} |a_{ik} - s_i^-|\})$$

Upper bound of the real eigenvalues different from 1 of a stochastic matrix in terms of the least off-diagonal element:

Theorem. Let $A = (a_{ik})_{1 \leq i,k \leq n}$ be a a **stochastic** matrix, and let s^+ and w be the least off-diagonal and diagonal entries of A, respectively. If λ is a real eigenvalue of A, then either $\lambda = 1$ (with algebraic multiplicity 1 if $s^+ > 0$) or $2w - 1 \leq \lambda \leq 1 - ns^+$.

J.M. P.: "Refining Gerschgorin disks through new criteria for nonsingularity" (2007). *Numerical Linear Algebra with Applications* **14**, pp. 665-671.

$\lambda \notin (\max_i \{a_{ii} - s_i^+ - \sum_{k \neq i} |a_{ik} - s_i^+|\} \quad \min_i \{a_{ii} - s_i^- + \sum_{k \neq i} |a_{ik} - s_i^-|\})$

$\bar{B}$-intervals: $\lambda \in \bigcup_{i=1}^n [a_{ii} - r_i^+ - \sum_{k \neq i} |r_i^+ - a_{ik}|\,,\ a_{ii} - r_i^- + \sum_{k \neq i} |r_i^- - a_{ik}|]$

EXAMPLE. $A = \begin{pmatrix} x+y & y & \cdots & \cdots & y \\ y & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & y \\ y & \cdots & \cdots & y & x+y \end{pmatrix}$

If $y > 0$:

- Gerschgorin: $[x - (n-2)y, x + ny]$
- $\bar{B}$-intervals: $[x, x + ny]$ $(r_i^+ = y, r_i^- = 0)$
- Exclusion interval: $(x, x + ny)$ $(s_i^+ = y, s_i^- = 0)$
- Eigenvalues: $x, x + ny$

Applications of bounds of the minimal eigenvalue of an M-matrix in linear progamming

M. García-Esnaola and J.M. P., *Sign consistent linear programing problems*. Optimization **58** (2009), 935–946.

A nonsingular matrix A with positive diagonal elements and nonpositive off-diagonal elements is an **M-matrix** if $A^{-1} \geq 0$.

R. Mathias and J. S. Pang, *Error bounds for the linear complementarity problem with a P-matrix*. Linear Algebra Appl. 132 (1990), 123–136.

X. Chen and S. Xiang, *Computation of error bounds for P-matrix linear complementarity problems*. Math. Program., Ser. A 106 (2006) 513–525.

X. Chen and S. Xiang, *Perturbation bounds of P-matrix linear complementarity problems*. SIAM J. Opt. 18 (2007), 1250–1265.

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The **linear complementarity problem** consists of finding vectors $x \in \mathbf{R}^n$ satisfying

$$Mx + q \geq 0, \quad x \geq 0, \quad x^T(Mx + q) = 0, \tag{1}$$

where M is an $n \times n$ real matrix and $q \in \mathbf{R}^n$. We denote this problem by $\mathrm{LCP}(M, q)$ and its solutions by x^*.

We say that a matrix is an ***H*-matrix** if its comparison matrix is a nonsingular M-matrix. An H-matrix with positive diagonals is a P-matrix.

If M in (1) is a P-matrix, then for any $x \in \mathbf{R}^n$:

$$\|x - x^*\|_\infty \leq \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty,$$

where I is the $n \times n$ identity matrix, D the diagonal matrix $D = \mathrm{diag}(d_i)$ with $0 \leq d_i \leq 1$ for all $i = 1, \ldots, n$, x^* is the solution of the $\mathrm{LCP}(M, q)$ and $r(x) := \min(x, Mx + q)$, where the min operator denotes the componentwise minimum of two vectors.

If M in (1) is an H-matrix with positive diagonals, then

$$\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty \leq \|\tilde{M}^{-1}\max(\Lambda, I)\|_\infty, \qquad (2)$$

where $\tilde{M}$ is the comparison matrix of M, Λ is the diagonal part of M ($\Lambda := \mathrm{diag}(m_{ii})$) and $\max(\Lambda, I) := \mathrm{diag}(\max\{m_{11}, 1\}, \ldots, \max\{m_{nn}, 1\})$.

M. García-Esnaola and J.M. P., Comparisons of error bounds for linear complementarity problems of H-matrices. To appear in Linear Algebra Appl.

Theorem. Let us assume that $M = (m_{ij})_{1\leq i,j\leq n}$ is an **H-matrix with positive diagonal entries**. Let $\bar{D} = \mathrm{diag}(\bar{d}_1, \ldots, \bar{d}_n), \bar{d}_i > 0$, for all $i = 1, \ldots, n$, be a diagonal matrix such that $M\bar{D}$ is strictly diagonally dominant by rows. For any $i = 1, \ldots, n$, let $\bar{\beta}_i := m_{ii}\bar{d}_i - \sum_{j\neq i}|m_{ij}|\bar{d}_j$. Then

$$\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty \leq \max\{\frac{\max_i\{\bar{d}_i\}}{\min_i\{\bar{\beta}_i\}}, \frac{\max_i\{\bar{d}_i\}}{\min_i\{\bar{d}_i\}}\}. \qquad (3)$$

With a particular choice of $\bar{D}$, then $\bar{\beta}_i = 1$ for all i in Theorem 2.1 and so formula (2.2) becomes

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \max\{\max_i\{\bar{d}_i\}, \frac{\max_i\{\bar{d}_i\}}{\min_i\{\bar{d}_i\}}\}. \qquad (4)$$

Example. Let $k > 2$ and

$$M = \begin{pmatrix} 2k & -k+1 \\ -2k+2 & k \end{pmatrix}.$$

Then for that choice, we have $\bar{d} = (1/2, 1)^T$ and so, the bound (4) is 2. On the other hand, $\tilde{M} = M$,

$$\tilde{M}^{-1} = \begin{pmatrix} \frac{k}{4k-2} & \frac{k-1}{4k-2} \\ \frac{k-1}{2k-1} & \frac{k}{2k-1} \end{pmatrix}, \quad \|\tilde{M}^{-1}\max(\Lambda, I)\|_\infty = \frac{3k^2-2k}{2k-1}.$$

Therefore the bound (2) can be arbitrarily large.

M. García-Esnaola and J.M. P.,*Error bounds for linear complementarity problems of* **B-matrices**. Applied Mathematics Letters **22**, 1071-1075.

Bounds for the minimal eigenvalue

$$e := (1, \ldots, 1)^T, \quad r := Ae, \quad r_{\max} := \max_i \{r_i\}, \quad r_{\min} := \min_i \{r_i\} (> 0),$$

Theorem. Let $A = (a_{ij})_{1 \le i,j \le n}$ be a **Z-matrix strictly diagonally dominant by rows with positive diagonal entries**. Then A has a positive eigenvalue $\lambda_{\min}$ with minimal absolute value among all its eigenvalues, and satisfies:

$$(0 <) r_{\min} \le \lambda_{\min} \le r_{\max}.$$

Theorem. Let $A = (a_{ij})_{1\le i,j\le n}$ be a ***Z*-matrix strictly diagonally dominant by rows with positive diagonal entries**. Then:

$$\frac{1}{r_{\max}} \le \|A^{-1}\|_\infty \le \frac{1}{r_{\min}}. \qquad (1)$$

Moreover, for any matrix norm $\|\cdot\|$, one has $(1/r_{\max}) \le \|A^{-1}\|$.

The upper bound of the right hand side of (1) was already provided by Varah for any Z-matrix strictly diagonally dominant by rows in

J. M. Varah, *A lower bound for the smallest singular value of a matrix*, Linear Algebra Appl. 11 (1975), 3–5

through $\tilde r_{\min}$ instead of $r_{\min}$:

$$\tilde r_{\max} := \max_i\{\tilde r_i\},\ \tilde r_{\min} := \min_i\{\tilde r_i\}(>0), \quad \tilde r_i := |a_{ii}| - \sum_{j\ne i}|a_{ij}|,\ i = 1,\dots,n.$$

The lower bound of the right hand side of (1) **does not hold** for strictly diagonally dominant matrices whose entries have arbitrary sign:

$$A = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 3/8 & -1/8 \\ 1/4 & 1/4 \end{pmatrix}, \quad \|A^{-1}\|_\infty = \tfrac{1}{2}.$$

Given a nonsingular $n \times n$ M-matrix A, there exists a positive diagonal matrix D such that AD is strictly diagonally dominant by rows (with positive diagonal entries). Given the matrix AD, let $r^D := (AD)e$, where is given in (2.1), and given $r^D = (r_1^D, \ldots, r_n^D)^T$ then we can define

$$r_{\max}^D := \max_i \{r_i^D\}, \quad r_{\min}^D := \min_i \{r_i^D\} (> 0).$$

R. S. Varga, *On diagonal dominance arguments for bounding* $\|A^{-1}\|_\infty$, Linear Algebra Appl. 14 (1976), 211–217:

$$\|A^{-1}\|_\infty \leq \frac{\max_i \{\ d_i\}}{r_{\min}^D}.$$

Theorem. Let $A = (a_{ij})_{1\le i,j\le n}$ be a nonsingular M-matrix and let $D = \mathrm{diag}(d_i)$ be a positive diagonal matrix such that AD is strictly diagonally dominant by rows. Then A has a positive eigenvalue $\lambda_{\min}$ with minimal absolute value among all its eigenvalues, and satisfies:

$$(0 <) \frac{r^D_{\min}}{\max_i\{d_i\}} \le \lambda_{\min} \le \frac{r^D_{\max}}{\min_i\{d_i\}}.$$

Besides, one has

$$\frac{\min_i\{d_i\}}{r^D_{\max}} \le \|A^{-1}\|_\infty \le \frac{\max_i\{\ d_i\}}{r^D_{\min}}.$$

J.M. P.: "Eigenvalue bounds for some classes of P-matrices". *Numerical Linear Algebra with Applications* **16** (2009), pp. 871-882..

Minimal eigenvalue of TP matrices

Given $i \in \{1, \ldots, n\}$ let

$$J_i := \{j \mid |j-i| \text{ is odd}\}, \quad K_i := \{j \neq i \mid |j-i| \text{ is even}\}.$$

Theorem. Let A be a nonsingular totally positive matrix, and let $\lambda_{\min}(>0)$ be its minimal eigenvalue. Then:

$$\lambda_{\min} \geq \min_i \{a_{ii} - \sum_{j \in J_i} a_{ij}\}. \tag{1}$$

Gerschgorin Theorem applied to any real matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ implies that

$$\min_i \{a_{ii} - \sum_{j \neq i} a_{ij}\} \leq \min_i \{\mathrm{Re}\lambda_i\}. \tag{2}$$

The following nonsingular matrix A is totally positive:

$$A = \begin{pmatrix} 12 & 7 & 1 \\ 0 & 6 & 1 \\ 0 & 3 & 8 \end{pmatrix}.$$

The eigenvalues of A are 12, 9 and 5. The bound given by (1) implies that $\lambda_{\min} \geq 5$ and so it **cannot be improved**. However, the lower bound for $\lambda_{\min}$ given by (2) is now $\lambda_{\min} \geq \min\{4, 5, 5\} = 4$.

J.M. P.: "Eigenvalue bounds for some classes of P-matrices". *Numerical Linear Algebra with Applications* **16** (2009), pp. 871-882..

An application to CAGD

Given a sequence of functions $u_0, \ldots, u_n$ on $[a, b]$ such that

$$\sum_{i=0}^{n} u_i(t) = 1 \quad \forall t \in [a, b]$$

(i.e., $(u_0, \ldots, u_n)$ is normalized) and a sequence of points in $\mathbf{R}^k$ $(C_0, \ldots, C_n)$, we may define a curve

$$\gamma(t) = \sum_{i=0}^{n} C_i u_i(t), \quad t \in [a, b].$$

The points C_i, $i = 0, \ldots, n$ are the *control points*.

The polygon $C_0 \cdots C_n$ is called the *control polygon* of the curve γ.

Definition. A normalized system of nonnegative functions $(u_0, \ldots, u_n)$ is called *blending*.

A system $(u_0, \ldots, u_n)$ is blending if and only if the generated curves always lie in the convex hull of the control polygon (*convex hull property*).

$(u_0, \ldots, u_n)$ basis of a vector space of functions.

Collocation matrices: for $a \leq t_0 < t_1 < \cdots < t_n \leq b$

$$M \begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{pmatrix} := (u_j(t_i))_{i,j=0,\ldots,n}$$

$(u_0, \ldots, u_n)$ is a system of blending functions if and only if all the collocation matrices are stochastic.

It is desirable that the curve "mimics" the control polygon and that the control polygon even "exaggerates" the shape of the curve.

Definition. A matrix is said to be *totally positive* (TP) if all its minors are nonnegative.

Definition. A system of functions $(u_0, \ldots, u_n)$ is *totally positive* (TP) if all its collocation matrices are totally positive.

TP systems of functions are interesting due to the *variation diminishing* properties of totally positive matrices

Definition. A TP basis $(u_0, \ldots, u_n)$ is *normalized totally positive* (NTP) if

$$\sum_{i=0}^{n} u_i(t) = 1, \quad \forall t \in I.$$

Collocation matrices of NTP systems are TP and stochastic

Endpoint interpolation property: the first control point always coincides with the start point of the curve and the last control point always coincides with the final point of the curve.

Theorem. *If a given basis satisfies simultaneously the variation diminishing, the endpoint interpolation and the convex hull properties then it is NTP.*

Definition. *A TP basis* $(b_0, \ldots, b_n)$ *of a space of functions* $\mathcal{U}$ *such that, for any TP basis* $(v_0, \ldots, v_n)$ *of* $\mathcal{U}$ *there exists a TP matrix* K *satisfying*

$$(v_0, \ldots, v_n) = (b_0, \ldots, b_n)K$$

is called a B-basis of $\mathcal{U}$.

Theorem. *Let* $\mathcal{U}$ *be a vector space of functions which has a totally positive basis. Then:*

(i) *There exists a B-basis* $\mathbf{b} = (b_0, \ldots, b_n)^T$ *of* $\mathcal{U}$.

(ii) *A basis of* $\mathcal{U}$ *is a B-basis of* $\mathcal{U}$ *if and only if it is of the form* $(d_0 b_0, \ldots, d_n b_n)$, *where* $d_k > 0$ *for all* k.

(iii) *A basis* $\mathbf{u}$ *of* $\mathcal{U}$ *is TP if and only if* $\mathbf{u}^T = \mathbf{b}^T K$ *and* K *is a nonsingular TP matrix.*

J.M. Carnicer and J.M. P., "Totally positive bases for shape preserving curve design and optimality of B-splines", *Computer Aided Geometric Design* **11** (1994), 633-654.

Theorem. *Let $\mathcal{U}$ be a vector space of functions which has a NTP basis. Then:*

(i) *There exists a unique normalized B-basis* $\mathbf{b}$ *of* $\mathcal{U}$.

(ii) *A basis* $\mathbf{u}$ *of* $\mathcal{U}$ *is NTP if and only if* $\mathbf{u}^T = \mathbf{b}^T K$ *and* K *is a nonsingular stochastic TP matrix.*

A normalized B-basis has optimal shape preserving and stability properties.

Theorem. *A basis is a normalized B-basis if and only if it satisfies the least variation diminishing, the endpoint interpolation and the convex hull properties simultaneously.*

Book on the subject: “Shape preserving representations in Computer-Aided Geometric Design” (J.M. P., editor). Nova Science Publishers, Commack (New York), 1999.

If a space does not possess normalized B-basis, then it does not possess any shape preserving representation

Examples of B-bases

(a) The Bernstein basis is a normalized B-basis of the space of polynomials of degree less than or equal to n on a compact interval $[a,b]$:

$$b_i(t) := \binom{n}{i} \left(\frac{t-a}{b-a}\right)^i \left(\frac{b-t}{b-a}\right)^{n-i}, \quad i = 0, \ldots, n.$$

(b) The monomial basis $(1, t, \ldots, t^n)$ is a B-basis of the space of polynomials of degree less than or equal to n on $(0, \infty)$.

(c) The B-spline basis is a normalized B-basis of the space of polynomial splines on a given interval with a prescribed sequence of knots.

(d) In the space of Müntz polynomials on $[0, +\infty)$

$$\mathcal{M} := \{\sum_{i=1}^{n} c_i t^{\lambda_i} \,|\, c_i \in \mathbf{R},\ i = 0, 1, \ldots, n\},$$

$\lambda_0 < \cdots < \lambda_n$, the generalized monomial basis given by $(t^{\lambda_0}, \ldots, t^{\lambda_n})$ is a B-basis.

Progressive iterative approximation

H. LIN, H. BAO, G. WANG (2005), Totally positive bases and progressive iteration approximation, Computer & Mathematics with Applications 50, 575-586.

$$\gamma(t) = \sum_{i=0}^{m} P_i \, u_i(t)$$

Now we parameterize the control points P_i with the real increasing sequence $t_0 < t_1 < \cdots < t_m$, where the parameter t_i is assigned to the control point P_i for $i = 0, 1, \ldots, m$. Then we have the starting curve

$$\gamma^0(t) = \sum_{i=0}^{m} P_i^0 \, u_i(t)$$

of the sequence where $P_i^0 = P_i$ for $i = 0, 1, \ldots, m$. The remaining curves of the sequence, $\gamma^{k+1}(t)$ for $k \geq 0$, can be calculated as follows:

$$\gamma^{k+1}(t) = \sum_{i=0}^{m} P_i^{k+1}\, u_i(t),$$

with $\Delta_i^k = P_i - \gamma^k(t_i)$ and $P_i^{k+1} = P_i^k + \Delta_i^k$ or $i = 0, 1, \ldots, m$. Then $\Delta_j^k = \Delta_j^{k-1} - \sum_{i=0}^{n} \Delta_i^{k-1} u_i(t_j)$, for $j = 0, 1, \ldots, m$.

The iterative process can be written in **matrix form** in the following way:

$$\left[\Delta_0^k, \Delta_1^k, \ldots, \Delta_m^k\right]^\top = (I - B)\, \left[\Delta_0^{k-1}, \Delta_1^{k-1}, \ldots, \Delta_m^{k-1}\right]^\top$$

where I is the identity matrix of $n+1$ order and B is the collocation matrix of the basis $(u_0, \ldots, u_m)$ at $t_0 < t_1 < \cdots < t_m$.

Proposition. Let A be a nonsingular $n \times n$ TP matrix. Then:

(i) All the eigenvalues of A are positive.

(ii) Given the $n \times n$ diagonal matrix $J := \operatorname{diag}(1, -1, 1, \ldots, (-1)^{n-1})$, the matrix $JA^{-1}J$ is TP.

Theorem. The progressive iterative approximation process converges for any nonsingular collocation matrix B of an NTP basis.

Key fact: $\rho(I - B) < 1$ (B has positive eigenvalues because it is totally positive)

Which are the bases with fastest convergence rates?

Theorem. Given a space U with an NTP basis, the normalized B-basis of U provides a progressive iterative approximation with the fastest convergence rates among all NTP bases of U.

Key facts of the proof: The **minimal eigenvalue** of the collocation matrix of the normalized B-basis is **maximum** among the minimal eigenvalues of the collocation matrices of any other NTP basis.

- U has a unique normalized B-basis $(b_0, \ldots, b_n)$. If $(v_0, \ldots, v_n)$ is another NTP basis of U, then there exists a stochastic TP matrix K such that

$$(v_0, \ldots, v_n) = (b_0, \ldots, b_n)K.$$

Optimal conditioning of Bernstein collocation matrices

DELGADO J., P. J.M.: "Optimal conditioning of Bernstein collocation matrices" (2009). *SIAM J. Matrix Anal. Appl.* **31**, 990-996..

DELGADO J., P. J.M.: "Running Relative Error for the Evaluation of Polynomials" (2009). *SIAM Journal on Scientific Computing* **31** , pp. 3905-3921.

Theorem. Let $(b_0, \ldots, b_n)$ be the **Bernstein basis**, let $(v_0, \ldots, v_n)$ be another NTP basis of P_n on $[0, 1]$, let $0 \leq t_0 < t_1 < \cdots < t_n \leq 1$ and $V := M \binom{v_0,\ldots,v_n}{t_0,\ldots,t_n}$ and $B := M \binom{b_0,\ldots,b_n}{t_0,\ldots,t_n}$. Then

$$\kappa_\infty(B) \leq \kappa_\infty(V).$$

$$\mathrm{Cond}(A) := \| \, |A^{-1}| \, |A| \, \|_\infty.$$

Theorem. Let $(b_0, \ldots, b_n)$ be the **Bernstein basis**, let $(v_0, \ldots, v_n)$ be another TP basis of P_n on $[0, 1]$, let $0 \leq t_0 < t_1 < \cdots < t_n \leq 1$ and $V := M\binom{v_0,\ldots,v_n}{t_0,\ldots,t_n}$ and $B := M\binom{b_0,\ldots,b_n}{t_0,\ldots,t_n}$. Then

$$\mathrm{Cond}(B^T) \leq \mathrm{Cond}(V^T).$$

A direct method to compute the eigenvalues of convexity preserving matrices

Theorem. Let A be an $n \times n$ TP matrix. Then all the eigenvalues of A are nonnegative. If A is STP, then they are also distinct (and positive).

DELGADO J., P. J.M.: "Computation of the eigenvalues of convexity preserving matrices" (2009). *Applied Mathematics Letters* **22**, pp. 470-474.

A vector $v = (v_1, v_2, \ldots, v_n)^T \in \mathbf{R}^n$ is said to be *k-convex* if $\Delta^k v_i \geq 0$ for all $i \in \{1, \ldots, n-k\}$, where

$$\Delta^k v_i := \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} v_{i+j}.$$

A matrix A is said to be *k-convexity preserving* if for any k-convex vector v, the vector Av is also k-convex.

Matrices that are r-convexity preserving for $r = 0, \ldots, k$ arise in many practical and theoretical problems. The case $k = 1$ corresponds to the important case of nonnegative matrices which are **monotonicity preserving**.

J.M. P.: "Hierarchical open Leontief models", *Linear Algebra and its Applications* **428** (2008), 2549-2559.

$$E := \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \ldots & 1 & 0 \\ 1 & \ldots & 1 & 1 \end{pmatrix}, \qquad E^{-1} = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ -1 & 1 & \ddots & & \vdots \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & -1 & 1 \end{pmatrix}.$$

$E_1 := E$ and for $j \geq 2$,

$$E_j := \left(\begin{array}{c|c} I_{j-1} & 0 \\ \hline 0 & E \end{array}\right)$$

Theorem. Let A be a r-convexity preserving matrix for $r = 0, 1, \ldots, k$ $(k \geq 1)$. Then:

$$(E_1 \cdots E_k)^{-1} A (E_1 \cdots E_k) = \left(\begin{array}{c|c} \Lambda_k & * \\ \hline 0 & A_k \end{array} \right)$$

where Λ_k is an upper triangular matrix whose diagonal elements $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k (\geq 0)$ are the largest eigenvalues of A, and A_k is a nonnegative matrix with $\rho(A_k) \leq \lambda_k$.

Explicit formulae to calculate the k largest eigenvalues of a matrix A r-convexity preserving for $r = 0, 1, \ldots, k$ are derived and a stable **direct method** of $O(kn^2)$ elementary operations to compute the k largest eigenvalues of an $n \times n$ matrix r-convexity preserving for all $r = 0, 1, \ldots, k$ is obtained.

A vector $v \in \mathbf{R}^n$ is said to be *k-concave* if the vector $-v$ is k-convex.

P_{k-1} will denote the set of k-convex and k-concave vectors.

Theorem. Let A be a nonsingular totally positive matrix such that $AP_r \subseteq P_r$, $r = 0, \ldots, n-1$. **Then** A is an r-convexity preserving matrix for all r and A is similar to an upper triangular matrix of the form:

$$(E_1 \cdots E_{n-1})^{-1} A (E_1 \cdots E_{n-1}) = \begin{pmatrix} \lambda_1 & \dots & \dots & \dots \\ 0 & \lambda_2 & \dots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

Richardson method for stochastic TP matrices

Let A be a nonsingular matrix. **Richardson iterative method** for solving the linear system $Ax = b$ can be written by the recurrence

$$x_{n+1} = x_n - Ax_n + b, \quad n = 0, 1, 2, \ldots$$

Denoting by y the new iteration obtained from x, the method can be expressed by the equation

$$y = (I - A)x + b,$$

where I denotes the identity matrix.

Richardson iterative methods always converges for nonsingular TP stochastic matrices.

Theorem. Let A be a nonsingular TP stochastic matrix. Then, the Richarson iterative method **converges** to the solution of the system $Ax = b$ and the convergence speed corresponds to $\rho(I - A) = 1 - \lambda_{\min}(A)$.

In order to **accelerate the convergence** of the method, it is usual to perform a relaxation of the method, replacing the role of x_{n+1} by $(1-w)x_n + wx_{n+1}$. In the case of Richardson method, this leads to the **modified Richardson method**

$$y = (I - wA)x + wb,$$

Theorem. Let A be a nonsingular TP stochastic matrix. Then, the modified Richarson iterative method **converges** to the solution of the system $Ax = b$ if and only if $w \in (0, 2)$. The **optimal** convergence speed corresponding to $\rho_{\text{opt}} = \rho(I - w_{\text{opt}}A)$ is achieved for

$$w_{\text{opt}} = \frac{2}{1 + \lambda_{\min}(A)}, \quad \rho_{\text{opt}} = \frac{1 - \lambda_{\min}(A)}{1 + \lambda_{\min}(A)}$$

where $\lambda_{\min(A)} > 0$ denotes the minimal eigenvalue of A.

An iterative method for general nonsingular TP matrices

For stochastic matrices we know in advance that $\rho(A) = 1$. We can always reduce our problem to the stochastic matrices by means of a **scaling**.

Let us introduce the vector $e := (1, \ldots, 1)^T$ and for any matrix A we introduce an associated diagonal matrix

$$D := \mathrm{diag}(d_1, \ldots, d_n), \quad (d_1, \ldots, d_n)^T = d := Ae.$$

If A is nonsingular nonnegative matrix, then D has positive diagonal entries.

The scaling leading to stochastic matrices is **optimal**

Richardson method is closely related with the Progressive iterative approximation of C.A.G.D.