Linear Algebra and Quantum Computing

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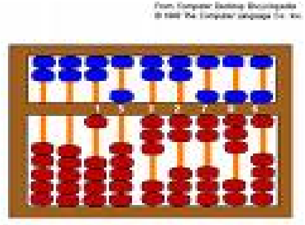
Joint work with Yiu-Tung Poon (Iowa State University).

General computing models

Input $\longrightarrow$ Computing Unit $\longrightarrow$ Output

Input ⟶ Computing Unit ⟶ Output

Classical computing (Abacus)



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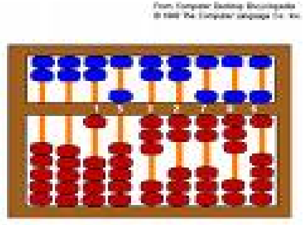

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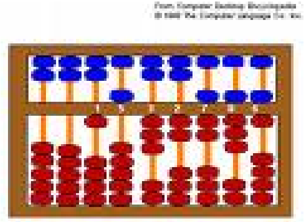

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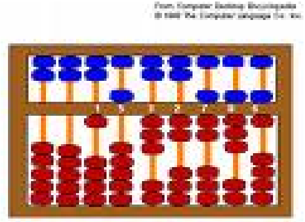

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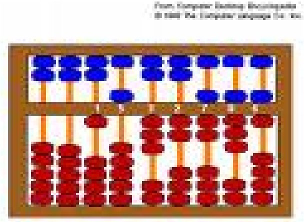

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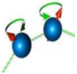
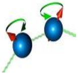
- Output - $(0, 1)$ sequences realized as visual images, which can be viewed or printed.

Quantum computing

 $\longrightarrow$ Quantum Computing Unit / Optical lattices, NMR $\longrightarrow$ 

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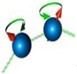

→ **Quantum Computing Unit**
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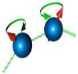

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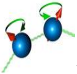
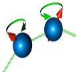
 → Quantum Computing Unit
Optical lattices, NMR → 

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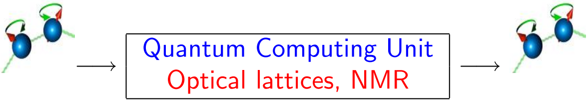

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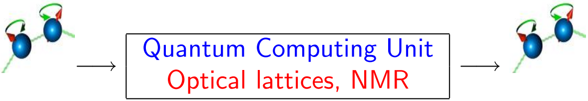

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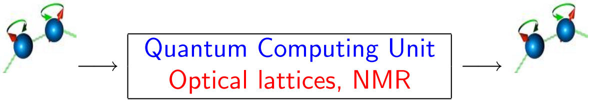

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- All these require the understanding of mathematics, physics, chemistry, computer sciences, engineering, etc.

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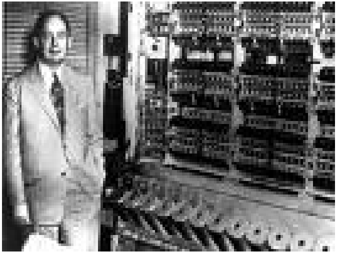

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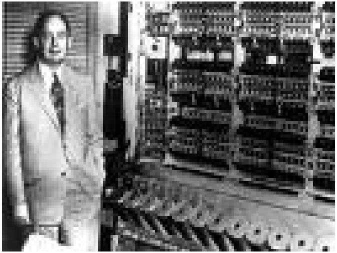

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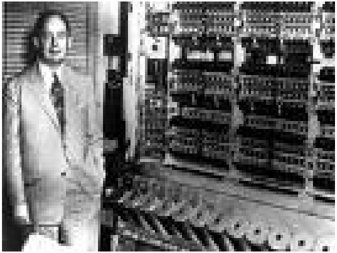

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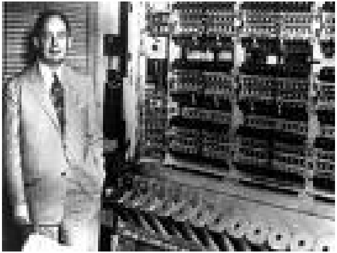

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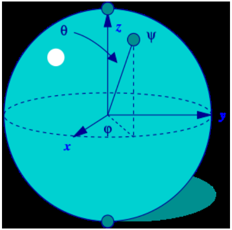
- One can apply a quantum operation to a state in superposition.

Matrix and Bloch sphere

- It is convenient to represent the quantum state $|\psi\rangle$ as a rank-one orthogonal projection:

$$|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

with $x, y, z \in \mathbb{R}$, $x^2 + y^2 + z^2 = 1$.



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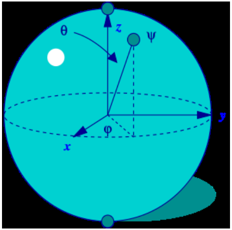

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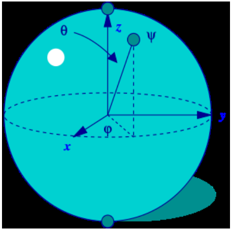

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- One can apply a single quantum operation to **ALL** the states $|x_1 \cdots x_k\rangle$ simultaneously if

$$|\psi\rangle = \sum_{x_i \in \{|\uparrow\rangle, |\downarrow\rangle\}} \gamma_{x_1 \cdots k} |x_1\rangle \otimes \cdots \otimes |x_k\rangle.$$

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- The theory was discovered way before the applications!

Man-Duen Choi

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A general problem

Since every quantum operation / channel is a trace preserving completely positive linear map, it is interesting to study the following.

Question

Given $A_1, \ldots, A_k \in M_n$ and $B_1, \ldots, B_k \in M_m$, is there a (unital/trace preserving) completely positive linear map $\mathcal{L}$ satisfying

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- Understand the duality relation between the trace preserving completely positive linear maps and the unital preserving completely positive linear maps.

Basic results

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It is known that $x \prec y$ if and only if there is a doubly stochastic matrix D such that $x = yD$.

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- The matrix D can be chosen so that the first k rows all equal and the last $n-k$ rows all equal.
- One can use D to construct $m \times n$ matrices $F_1, \ldots, F_r$ with $r = \max(m, n)$ such that

$$B = \sum_{j=1}^{r} F_j A F_j^* \quad \text{and} \quad \sum_{j=1}^{r} F_j^* F_j = I_n.$$

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- $\lambda(B) \prec (a_+, 0, \ldots, 0, a_-)$ in $\mathbb{R}^{1\times m}$.
- There is an $n \times m$ row stochastic matrix (nonnegative matrix with all row sums equal to one) D such that $\lambda(B) = \lambda(A)D$.

- The matrix D can be chosen so that the first k rows all equal and the last $n-k$ rows all equal.
- One can use D to construct $m \times n$ matrices $F_1, \ldots, F_r$ with $r = \max(m, n)$ such that
$$B = \sum_{j=1}^{r} F_j A F_j^* \quad \text{and} \quad \sum_{j=1}^{r} F_j^* F_j = I_n.$$
- For density matrices A and B, the condition trivially holds.

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Question

Can we deduce this result from the previous one using duality of completely positive linear map?

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Reason: If there were such a map, it would send A_1 to B_1 for

$$A_1 = A - I_4 = \text{diag}\,(3, 0, 0, -1) \quad \text{and} \quad B_1 = B - I_4 = \text{diag}\,(2, 2, -1, -1),$$

which is a contradiction.

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- B is in the convex hull of the unitary orbit $\mathcal{U}(A)$ of A:

$$\{UAU^* : U \text{ unitary}\}.$$

Results and questions on multiple matrices

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- Evidently, the result can be applied to commuting families $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$.
- The problem reduces to joint/multivariate majorization.
- What about non-commuting families?

For given $A \in H_n$, $B \in H_m$, and a postive integer r, we are interested in constructing/finding a CP map $\mathcal{L} : M_n \to M_m$ of the form $\mathcal{L}(A) = \sum_{j=1}^{r} F_j A F_j^*$ such that $\mathcal{L}(A) = B$.

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- The number of positive (negative) eigenvalues of $A \otimes I_r$ is more than or equal to the number of positive (negative) eigenvalues of B.

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- The eigenvalues of $A \otimes I_r$ interlace those of B, i.e.,

$$\lambda_j(A) \geq \lambda_{(j-1)r+1}(B) \text{ whenever } (j-1)r+1 \leq m, \quad \text{and}$$

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Remark Even if $A, B \in H_n$ are density matrices, the roles of A and B are not symmetric in the last two theorems.

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Remark Conditions (b) and (c) can be expressed in terms of eigenvalue inequalities using Littlewood-Richardson rules or the Horn sequences:

$$\sum_{j \in J_0} \lambda_j(B) \leq \sum_{j \in J_1} \lambda_j(C_1) + \cdots + \sum_{j \in J_r} \lambda_j(C_r)$$

with $(J_0, J_1, \dots, J_r) \in \mathcal{S}(n, r, \ell)$ for $\ell \in \{1, \dots, n-1\}$.

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Let $A, B \in H_n$ be positive semidefinite. Then $B = \mathcal{L}(A)$ for some TPCP map $\mathcal{L}$ of with rank r if and only if $A = \widetilde{\mathcal{L}}(B)$ for some TPCP map $\widetilde{\mathcal{L}}$ with rank r.

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Remark This is not true for unital completely positive linear maps.

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Derive numerical scheme (using gradient flow, positive semi-definite programming, etc.) to solve the following:
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We may impose additional conditons such as:

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$$\mathcal{L}(A_j) = B_j, \qquad j = 1, \ldots, k,$$

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We may impose additional conditons such as:

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- The sum of $r \times r$ principal submatrix of $\mathcal{L}$: $S_r(\mathcal{L}) = 0$ for a given r ($\mathcal{L}$ has rank less than r).

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Thank you for your attention!