Dimension Reduction For Damping Optimization In Linear Vibrating Systems

Peter Benner [1] **Zoran Tomljanović** [2] Ninoslav Truhar[2]

[1]Faculty of Mathematics, *Chemnitz University of Technology*, Chemnitz, Germany

[2]Odjel za matematiku, *University J. J. Strossmayer in Osijek*

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$$M\ddot{x} + D\dot{x} + Kx = 0,$$
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Quadratic eigenvalue problem:

$$(\lambda^2 M + \lambda D + K)x = 0.$$

Linearization

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- Let Φ simultaneously diagonalize pair M and K

$$\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I,$$

$\omega_1 \geq \omega_2 \geq \ldots \geq \omega_n$ are the eigenvalues of the undamped system.

With $x = \Phi x_\Phi$ and $y_1 = \Omega x_\Phi$, $y_2 = \dot{x}_\Phi$ we have

$$\frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D \Phi \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y},$$

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$$\dot{y} = Ay, \quad \text{with solution} \quad y = e^{At} y_0, \quad \text{where } y_0 \text{ is initial data.}$$

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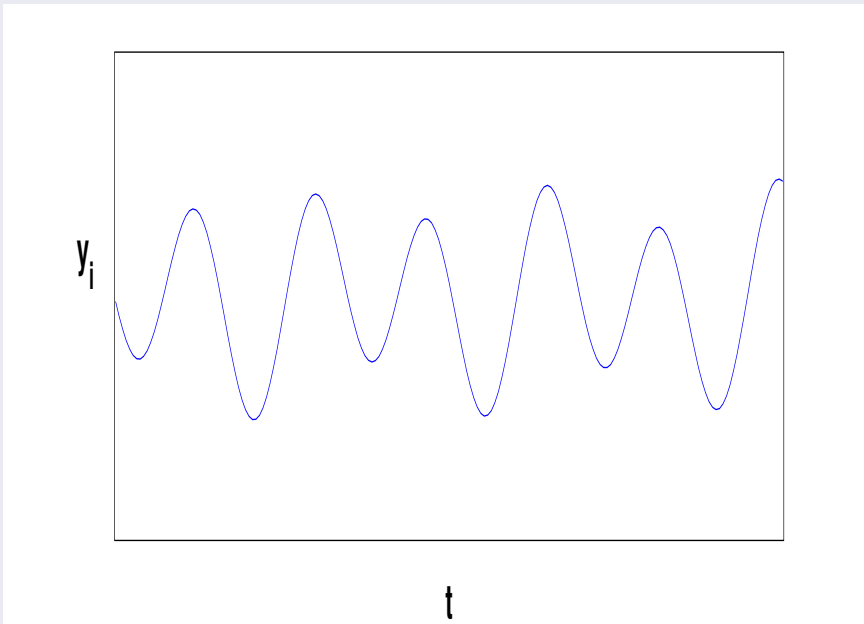

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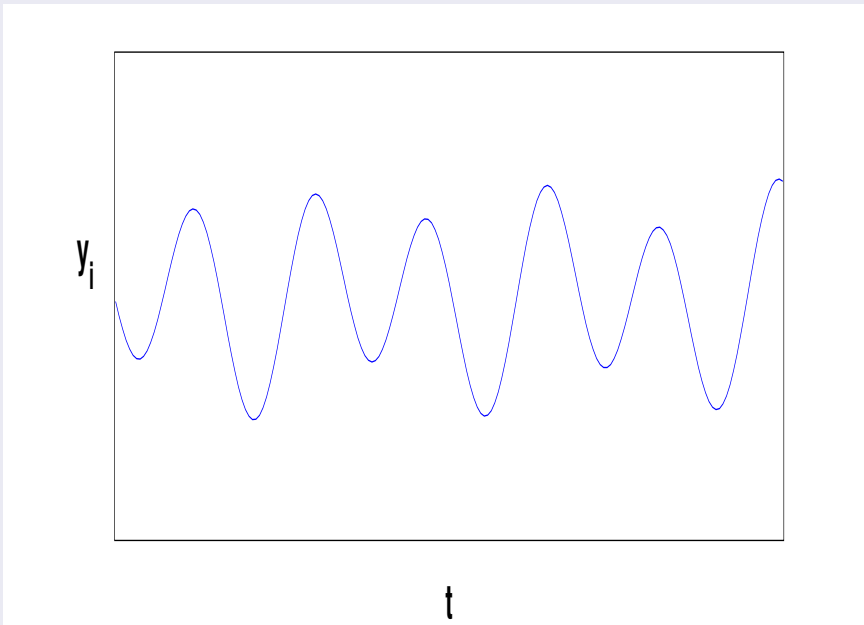

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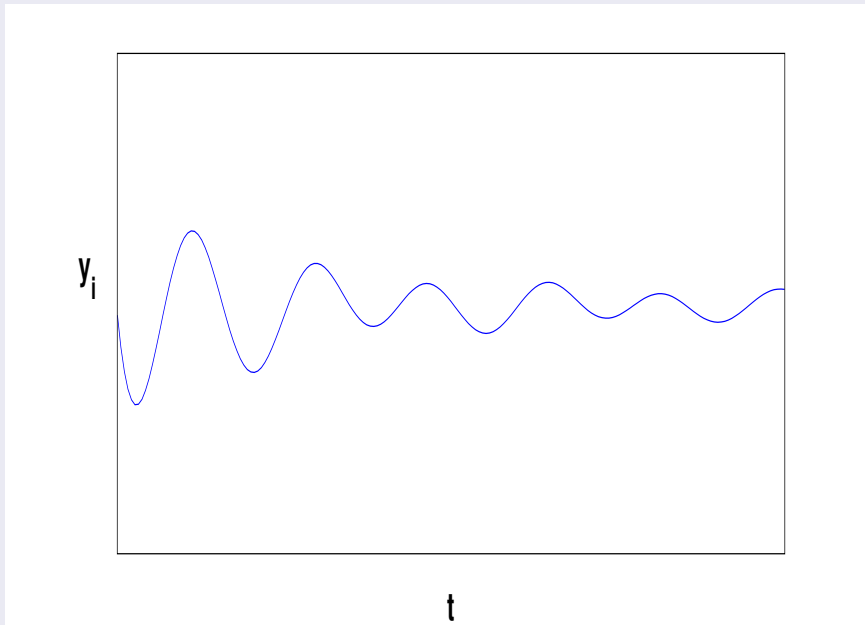

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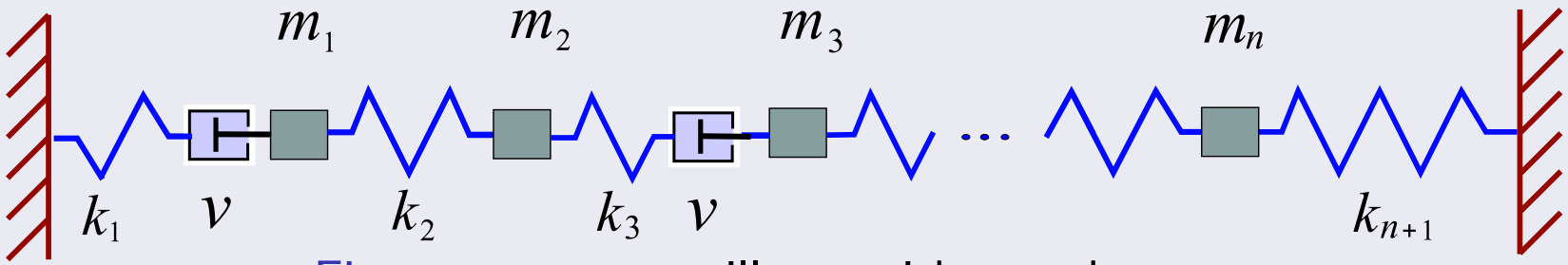

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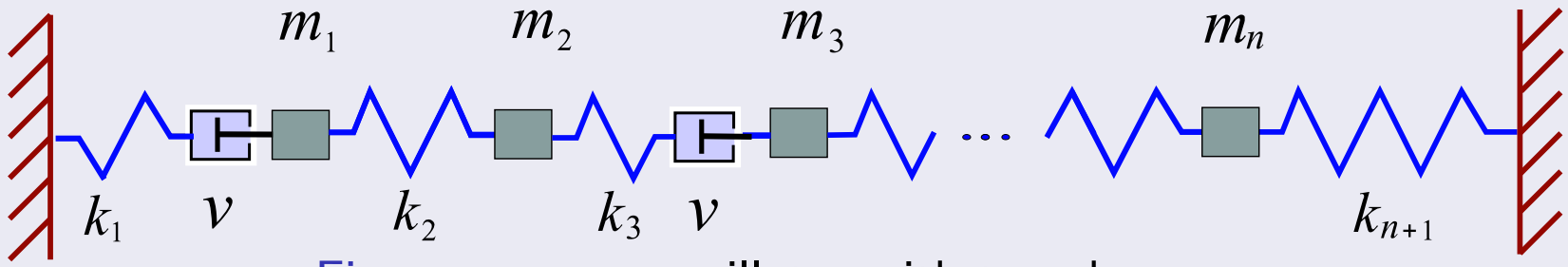

Figure: n-mass oscillator with two dampers

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n), \quad D \equiv C_{int} + C_{ext} = C_{int} + ve_1 e_1^T + ve_3 e_3^T.$$

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix},$$

$m_i > 0$ - masses, $k_i > 0$ - stiffness, v - viscosity.

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- We will use the following criterion:

$$\operatorname{tr} X \quad \rightarrow \quad min,$$

where X is the solution of

$$AX + XA^T = -Z.$$

A is $2n \times 2n$ matrix obtained from M, C, K; Z has structure.

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In example:

$$C_{ext} = ve_i e_i^T + ve_j e_j^T,$$

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For example if $n = 1000$:

discrete optimization over 500 000 different positions.

Efficient overall algorithm for dampers' optimization is still needed !

$\rightsquigarrow$ for determination of the optimal position and viscosity we need to solve 10 000 000 Lyapunov equation of dimension 2000.

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where

$$A_P = A_0 + P^T \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} P, \quad C = \Phi^T C_{ext} \Phi,$$

$$A_0 = \hat{A}_1 \oplus \hat{A}_2 \oplus \cdots \oplus \hat{A}_n, \quad \hat{A}_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -\alpha\omega_i \end{bmatrix}.$$

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$$\operatorname{tr} X = \operatorname{tr} X_P = (\frac{2}{\alpha} + \frac{\alpha}{2}) \sum_{i=1}^{n} \frac{1}{\omega_i}.$$

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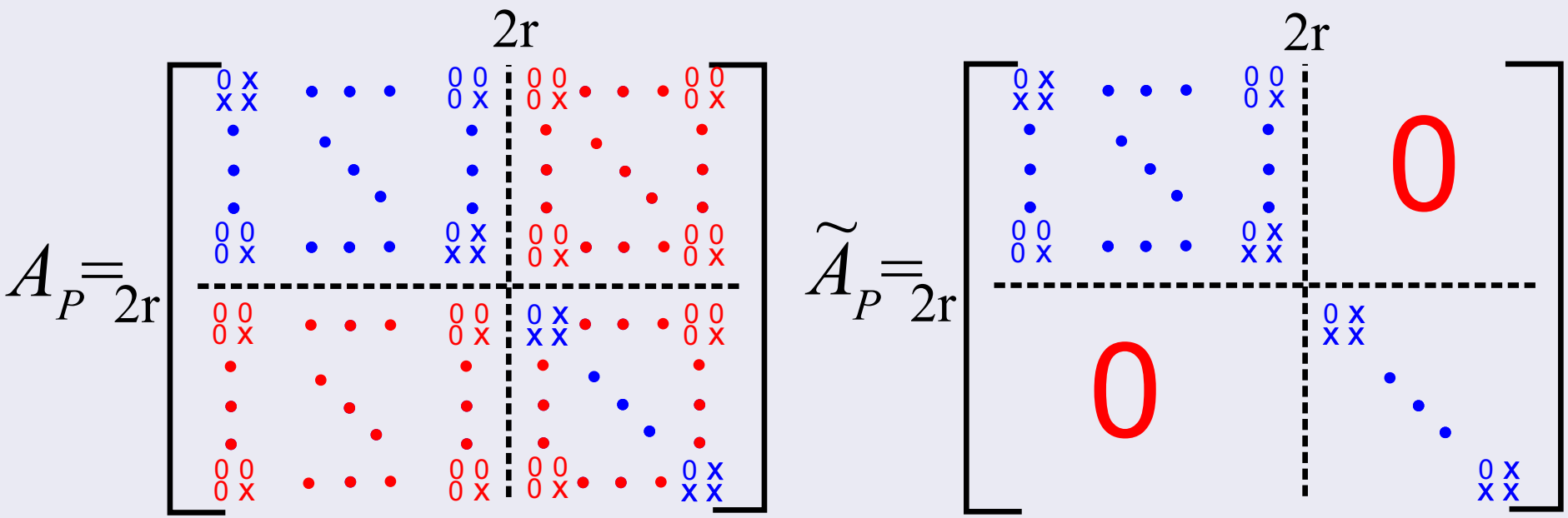

For $\|C(:, r+1:n)\| (= \|C(r+1:n, :)\|)$ small, we approximate A_p with

$$\widetilde{A}_P = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & \widetilde{A}_{22} \end{bmatrix}, \quad \text{where} \quad \widetilde{A}_{11} = A_p(1:2r, 1:2r)$$

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Approximation of the trace

$$\operatorname{tr} X \approx \operatorname{tr} \widetilde{X}_r + (\frac{2}{\alpha} + \frac{\alpha}{2}) \sum_{i=r+1}^{n} \frac{1}{\omega_i}.$$

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All previous formulas hold and we have the similar trace approximation.

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5. *Calculate the* $\mathrm{tr}\, X_r$, *where*

$$A_r X_r + X_r A_r^T = -I, \quad A_r = \begin{bmatrix} 0 & \Omega_r \\ -\Omega_r & -\alpha\Omega_r - C_r \end{bmatrix}.$$

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3. $\Omega_r = \mathrm{diag}(\omega_{p(1)}, \omega_{p(2)}, \ldots, \omega_{p(r)})$,
4. $C_r \in \mathbb{R}^{r \times r}$ *is matrix obtained from matrix* C *such that we take intersection of* $p(1), p(2), \ldots, p(r)$ *rows and columns from matrix* C,
5. *Calculate the* $\mathrm{tr}\, X_r$, *where*

$$A_r X_r + X_r A_r^T = -I, \quad A_r = \begin{bmatrix} 0 & \Omega_r \\ -\Omega_r & -\alpha\Omega_r - C_r \end{bmatrix}.$$

6. $\mathrm{tr}\, X = \mathrm{tr}\, X_r + (\frac{2}{\alpha} + \frac{\alpha}{2}) \sum_{i=1}^{n-r} \frac{1}{\omega_{\overline{p}(i)}}$.

Algorithm (for the trace approximation)

Input: C, α, *tol*, Φ *s.t.* $\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$, $\Phi^T M \Phi = I$,
Output: $\mathrm{tr}\, X(C)$

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Note: in viscosity optimization in step 1 and 2 norms should be calculated just once, e.g. $v = 1$.

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$$\begin{bmatrix} A_{11} & \varepsilon E \\ \varepsilon E^T & A_{22} \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & \varepsilon E \\ \varepsilon E^T & A_{22}^T \end{bmatrix} = -I, \quad \|E\|_F = 1.$$

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Apply standard perturbation bound for Sylvester equation (N. Higham).

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With $A_{11} = A_r$, where A_r is given in Algorithm for the trace approximation

$\tilde{A}_{22} = \hat{A}_{\overline{p}(1)} \oplus \cdots \oplus \hat{A}_{\overline{p}(n-r)}$,

$\tilde{X}_{ii}$ is approximations of X_{ii}; $X_{ii} = \tilde{X}_{ii} + \Delta X_{ii}$, $i = 1, 2$

The error matrices ΔX_{ii} satisfy

$$\begin{aligned} \frac{\|\Delta X_{11}\|_F}{\|X_{11}\|_F + \|X_{22}\|_F} &\leq 6 \frac{1}{\operatorname{sep}(A_{11}, -\tilde{A}_{22}^T)} \frac{1}{\operatorname{sep}(A_{11}, -A_{11}^T)} \varepsilon^2, \\ \frac{\|\Delta X_{22}\|_F}{\|\tilde{X}_{22}\|_F} &\leq \frac{\sqrt{3}}{\operatorname{sep}(\tilde{A}_{22}, -\tilde{A}_{22}^T)} \left(2 + \frac{\sqrt{n}}{\|\tilde{A}_{22}\|_F \|\tilde{X}_{22}\|_F}\right) \varepsilon, \end{aligned}$$

$\varepsilon = \|C(p, \overline{p})\|_F$ and the vectors $\overline{p}$ and p are calculated in Algorithm for the trace approximation.

Numerical Example

Consider *n-mass oscillator*

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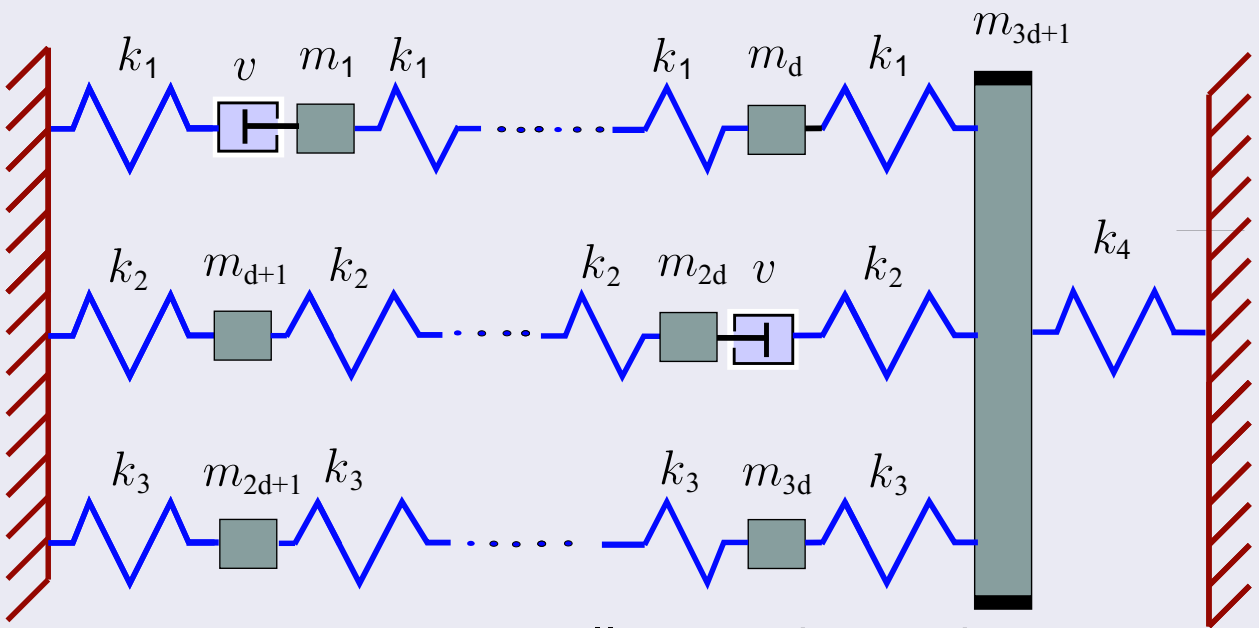

Figure: n-mass oscillator with two dampers

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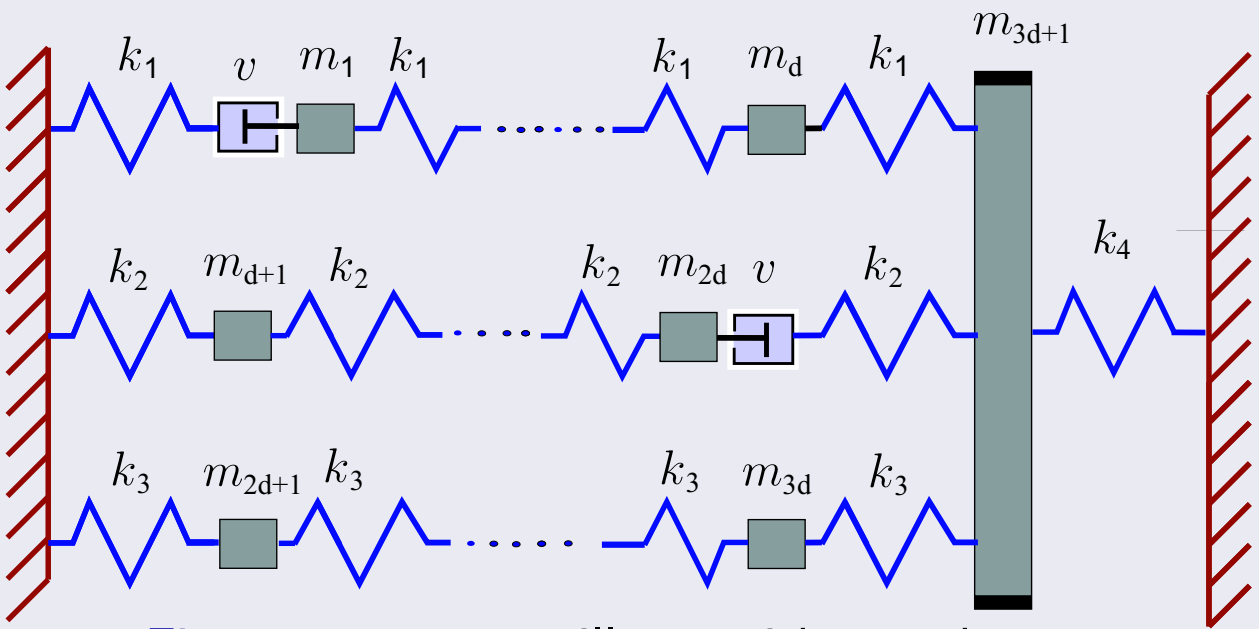

Figure: n-mass oscillator with two dampers

Mass matrix is

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n), \quad m_k = k, \quad k = 1, \ldots, n$$

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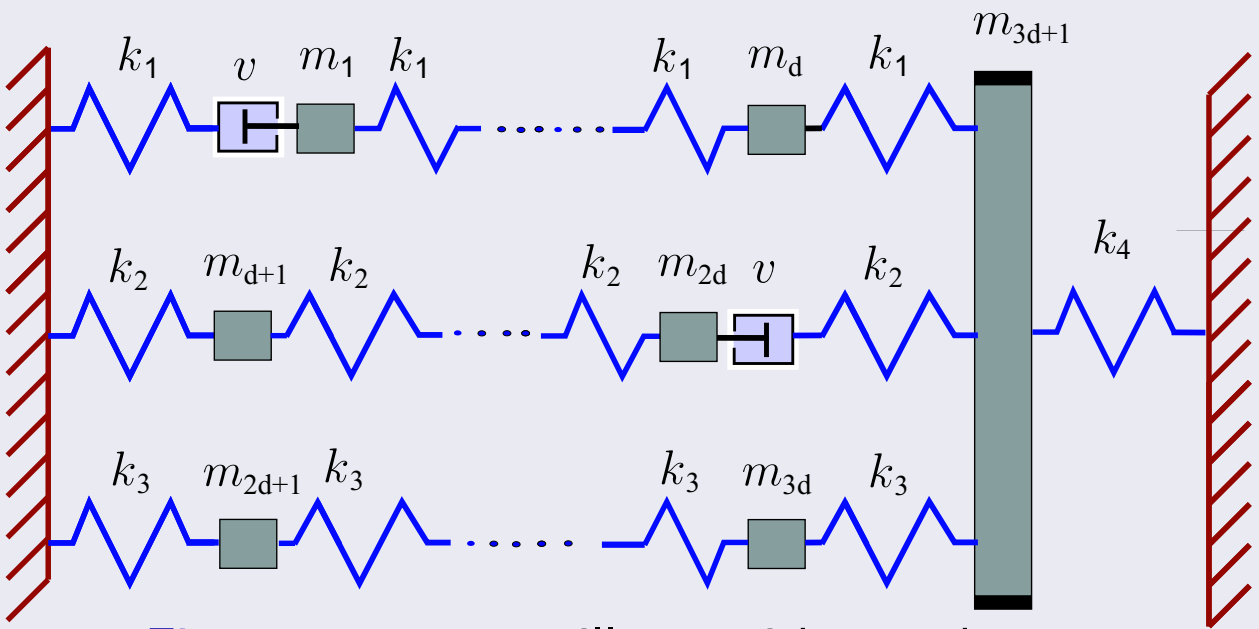

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$$d = 500, \quad n = 3d + 1 = 1501.$$

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$$K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \kappa_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_i \end{bmatrix}, \quad i = 1, 2, 3.$$

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$$k_1 = 1, \quad k_2 = 50 \quad k_3 = 100, \quad k_4 = 200$$

$$D = C_{int} + C_{ext} = 0.01 \cdot M^{1/2} \sqrt{M^{-1/2} K M^{-1/2}} M^{1/2} + ve_i e_i^T + ve_j e_j^T$$

$i = 1 : 70 : n, \quad j = i + 5 : 70 : n$ which gives 253 different positions.

Relative error for trace approximation and viscosity

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Optimal viscosity obtained with Matlab's function $\mathrm{fminbnd}$ $tol = 1e-4$.

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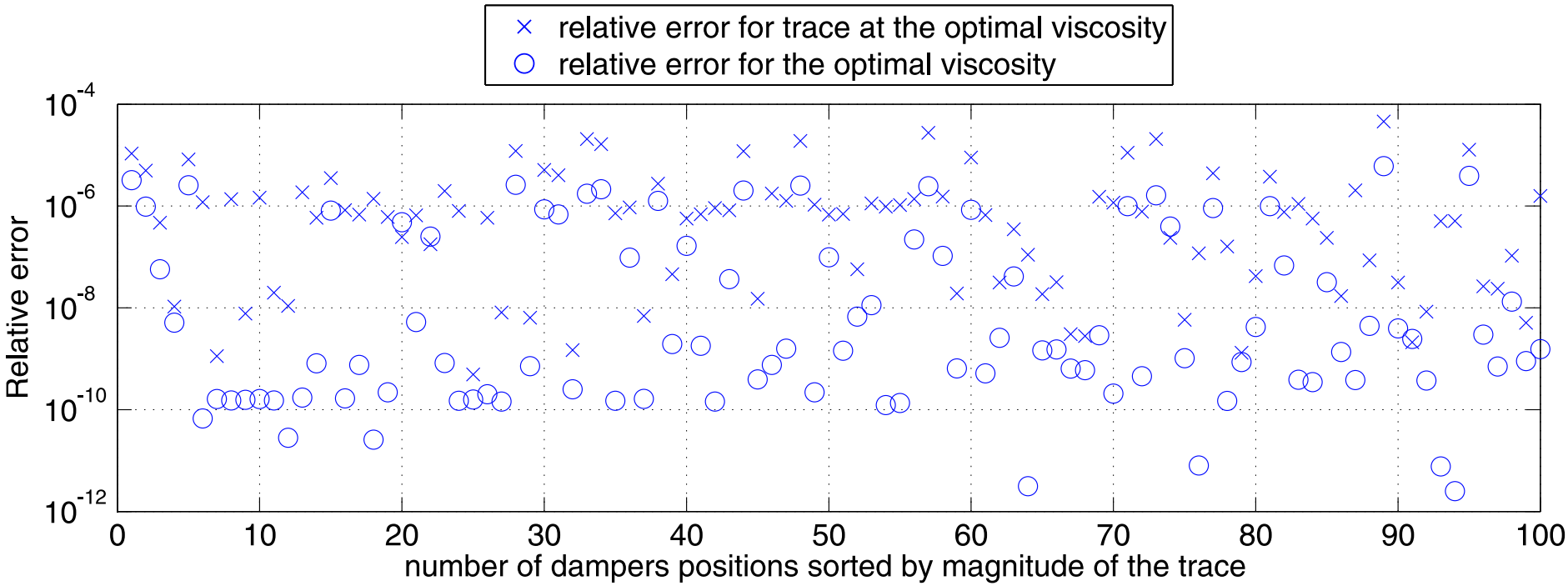

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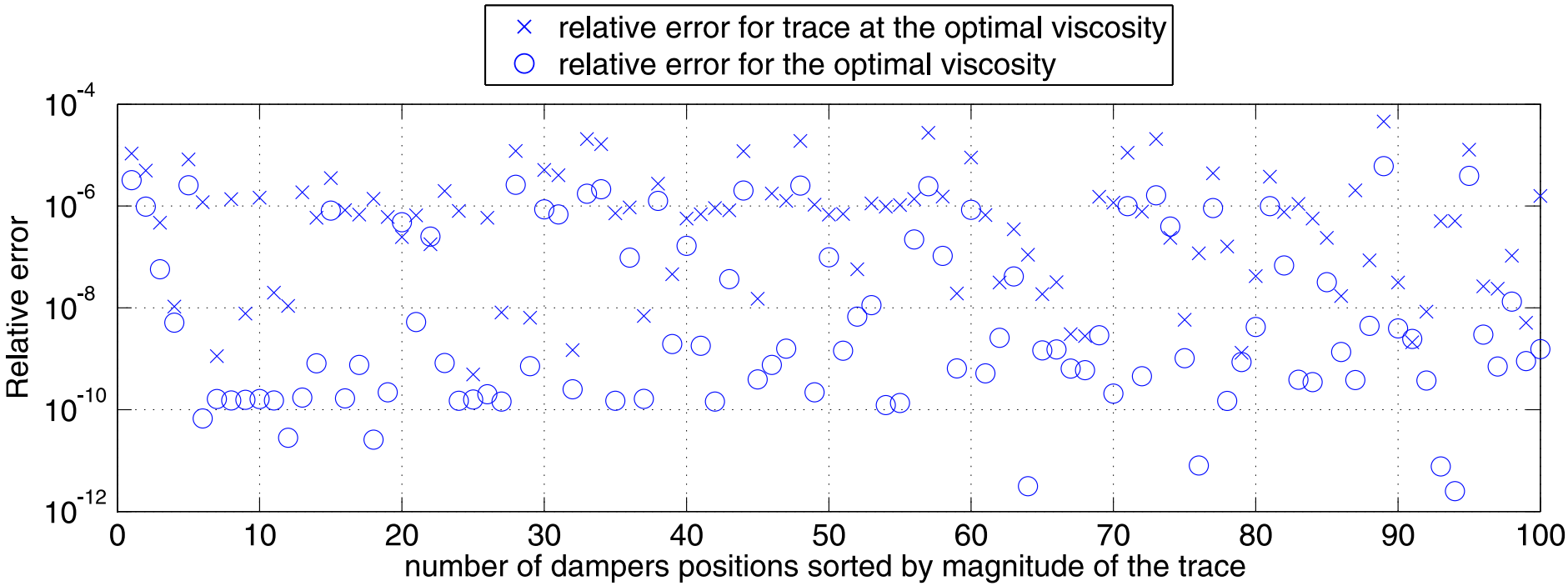

Optimal position is $i = 211, \quad j = 426$.

With both Algorithms $\rightsquigarrow$ the same optimal position!

($\mathrm{tr}(X(C_{opt})) = 2990313.07995$, $v_{opt} = 32.75013$).

Numerical Example

Dimension of reduction and time ratio

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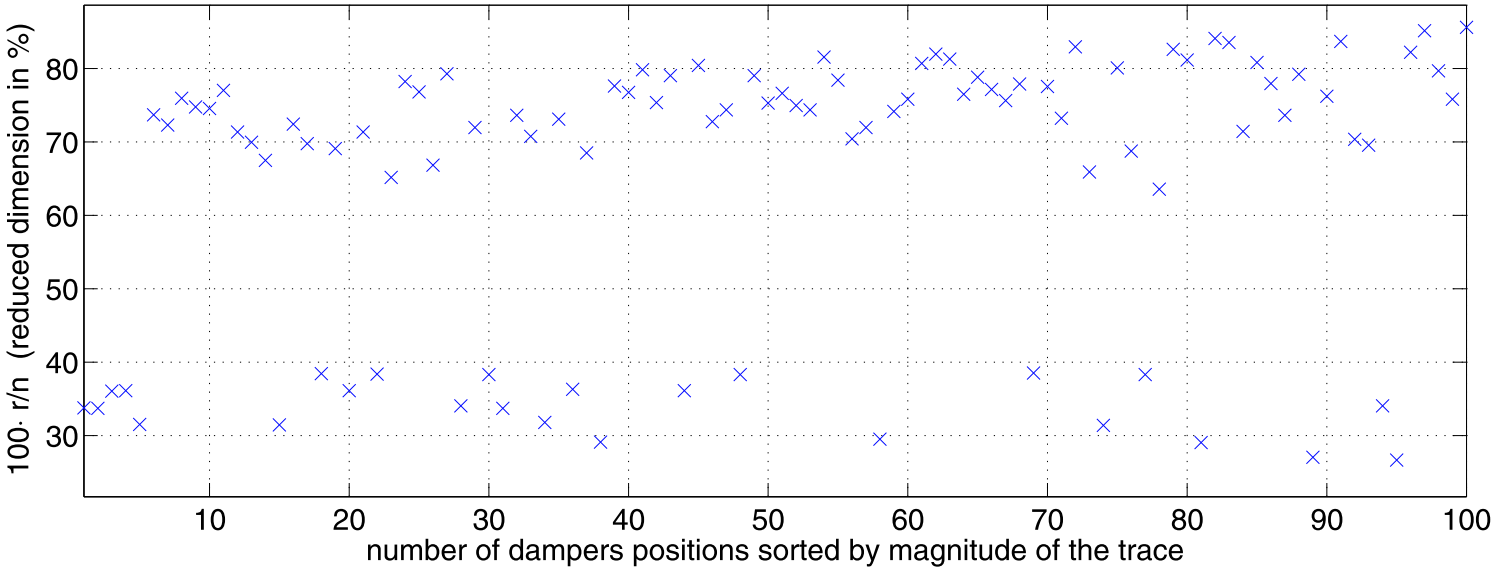

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