A permuted factors approach for the linearization of polynomial matrices

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25 May 2010

Introduction

Linearization is the transformation of a polynomial matrix $T(s) = T_n s^n + T_{n-1} s^{n-1} + ... + T_0$, $T_i \in \mathbb{C}^{p \times p}$ to a corresponding matrix pencil $L(s) = sL_1 - L_0$, $L_i \in \mathbb{C}^{np \times np}$ having the same finite divisor structure.

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- (+) Reliable numerical algorithms are available for matrix pencils.
- (+) Special techniques exist for structured matrix pencils (symmetric, Hamiltonian etc.).

Products of elementary matrices

The most common linearizations of $T(s)$ are the well known first and second companion linearizations $P(s)$ and $\hat{P}(s)$

$$P(s) = s\begin{bmatrix} I_p & 0 & \cdots & 0 \\ 0 & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & T_n \end{bmatrix} - \begin{bmatrix} 0 & I_p & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_p \\ -T_0 & -T_1 & \cdots & -T_{n-1} \end{bmatrix},$$

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- (+) $P(s)$, $\hat{P}(s)$ can be constructed by inspection of the coefficient matrices of $T(s)$.
- (+) The matrices involved are relatively sparse.
- (-) In case the matrix $T(s)$ exhibits symmetries, these are not reflected on $P(s)$ and $\hat{P}(s)$.

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- The present work extends the permuted factors approach focusing on
  - The construction of (companion like) linearizations using the unperturbed coefficients of $T(s)$.
  - Controlling certain aspects of the structure of the resulting pencils in order to preserve selected structural properties of the polynomial matrix.

Elementary matrices

Definition (Elementary matrices)

Let $T(s)$ a regular polynomial matrix. Then we define the following elementary matrices corresponding to $T(s)$ as follows:

$$A_k = \begin{bmatrix} I_{p(k-1)} & 0 & \cdots \\ 0 & C_k & \ddots \\ \vdots & \ddots & I_{p(n-k-1)} \end{bmatrix}, \; k = 1, 2, \ldots, n-1,$$

where

$$C_k = \begin{bmatrix} 0 & I_p \\ I_p & -T_k \end{bmatrix}$$

and

$$A_0 = diag\{-T_0, I_{p(n-1)}\}.$$

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Let $\mathcal{I} = (i_1, i_2, \ldots, i_m)$ be an ordered tuple containing indices from $\{0, 1, 2, \ldots, n-1\}$. Then $A_{\mathcal{I}} := A_{i_1} A_{i_2} \cdots A_{i_m}$.

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Theorem

[Antoniou and Vologiannidis, 2004] Let $i, j \in \{0, 1, 2, \ldots, n-1\}$. Then $A_i A_j = A_j A_i$ if and only if $|i-j| \neq 1$.

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Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two tuples. $\mathcal{I}_1$ will be termed equivalent to $\mathcal{I}_2$ ($\mathcal{I}_1 \sim \mathcal{I}_2$) if and only if $A_{\mathcal{I}_1} = A_{\mathcal{I}_2}$.

Products of elementary matrices

- We will use the following notation:
  - Let $k, l \in \mathbb{Z}$ with $k \leq l$. Then $(k:l) := \begin{cases} (k, k+1, ..., l), k \leq l \\ \varnothing, k > l \end{cases}$
  - $A_{\varnothing} = I_{np}$.
  - If $\mathcal{I} = (i_1, i_2, \ldots, i_m)$ then $\bar{\mathcal{I}} = (i_m, i_{m-1}, \ldots, i_1)$.

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A product of elementary matrices $A_{\mathcal{I}}$ will be termed **operation free** iff the block elements of $A_{\mathcal{I}}$ are either $0, I_p$ or $-T_i$ (for generic matrices T_i).

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Example

$$A_0 A_1 A_0 = \begin{bmatrix} 0 & -T_0 & \\ -T_0 & -T_1 & \\ & & I_{p(n-2)} \end{bmatrix} \text{ is operation free,}$$

$$A_1 A_0 A_1 = \begin{bmatrix} I_p & -T_1 & \\ -T_1 & T_1^2 - T_0 & \\ & & I_{p(n-2)} \end{bmatrix} \text{ is not.}$$

Products of elementary matrices

Lemma (Range Product)

The product $A_{(k:l)}$ is of the form

$$A_{(k:l)} = \begin{cases} \left[\begin{array}{c|c|c|c} I_{k-1} & & & \\ \hline & 0_{1\times(l-k+1)} & I & \\ \cline{2-3} & I_{l-k+1} & \begin{array}{c}-T_k\\ \vdots \\ -T_l\end{array} & \\ \hline & & & I_{n-l-1}\end{array}\right], & k>0 \\ \left[\begin{array}{c|c|c} 0_{1\times l} & -T_0 & \\ \hline I_l & \begin{array}{c}-T_1\\ \vdots\\ -T_l\end{array} & \\ \hline & & I_{n-l-1}\end{array}\right], & k=0\end{cases}, \quad k\le l\le n-1$$

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Definition (Standard forms)

- Column standard form: $\prod_{i=n-1}^{0} A_{(c_i:i)}$, where $c_i \in (0:i) \cup \{\infty\}$.

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Consider $(A_0A_1A_2A_3)(A_0A_1A_2)(A_0A_1)(A_0) = \begin{bmatrix} 0 & 0 & 0 & -T_0 \\ 0 & 0 & -T_0 & -T_1 \\ 0 & -T_0 & -T_1 & -T_2 \\ -T_0 & -T_1 & -T_2 & -T_3 \end{bmatrix}$.

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Theorem

Column and row standard forms are operation free.

Products of elementary matrices

Definition (Successor Infixed Property)

Let $\mathcal{I} = (i_1, i_2, \ldots, i_k)$ be an index tuple. $\mathcal{I}$ will be called successor infixed if and only if for every pair of indices $i_a, i_b \in \mathcal{I}$, with $1 \leq a < b \leq k$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$, such that $a < c < b$.

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- We can arrive to similar results using inverses of elementary matrices i.e.

$$A_{-k} := A_k^{-1} = \begin{bmatrix} I_{p(k-1)} & 0 & \cdots \\ 0 & C_k^{-1} & \ddots \\ \vdots & \ddots & I_{p(n-k-1)} \end{bmatrix}, \; k = 1, \ldots, n-1$$

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Example

$$(A_{-4}A_{-3}A_{-2}A_{-1})(A_{-4}A_{-3}A_{-2})(A_{-4}A_{-3})(A_{-4}) = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \\ T_2 & T_3 & T_4 & 0 \\ T_3 & T_4 & 0 & 0 \\ T_4 & 0 & 0 & 0 \end{bmatrix}.$$

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- *Then the matrix pencil*

$$sA_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{N}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})} - A_{(\mathcal{L}_{\mathcal{N}}, \mathcal{L}_{\mathcal{P}}, \mathcal{P}, \mathcal{R}_{\mathcal{P}}, \mathcal{R}_{\mathcal{N}})}$$

is a ***linearization*** *of $T(s)$* ***and its coefficients are operation free matrices****.*

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- *Let $\mathcal{N}$ be a permutation of the tuple $(-n : -k)$ and $\mathcal{L}_\mathcal{N}, \mathcal{R}_\mathcal{N}$ tuples with elements from $(-n : -k-1)$ s.t. $(\mathcal{L}_\mathcal{N}, \mathcal{N}, \mathcal{R}_\mathcal{N})$ satisfies the SIP.*
- *Then the matrix pencil*

$$sA_{(\mathcal{L}_\mathcal{N},\mathcal{L}_\mathcal{P},\mathcal{N},\mathcal{R}_\mathcal{P},\mathcal{R}_\mathcal{N})} - A_{(\mathcal{L}_\mathcal{N},\mathcal{L}_\mathcal{P},\mathcal{P},\mathcal{R}_\mathcal{P},\mathcal{R}_\mathcal{N})}$$

is a ***linearization*** *of $T(s)$* ***and its coefficients are operation free matrices****.*

Theorem (Linearizations of polynomial matrices)

- *Let $T(s)$ be a polynomial matrix of degree n with T_0, T_n nonsingular.*
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Lemma

A product $A_\mathcal{I}$ is block symmetric if-f $\mathcal{I} \sim \bar{\mathcal{I}}$.

- Notice that T_0 is allowed to be singular if $0 \notin (\mathcal{L}_\mathcal{P}, \mathcal{R}_\mathcal{P})$ while the same holds for T_n if $-n \notin (\mathcal{L}_\mathcal{N}, \mathcal{R}_\mathcal{N})$.

Corollary (Lancaster's symmetric linearizations)

The symmetric linearizations $L_k(s) = sS_{k-1} - S_k, k = 1, \ldots, n$ of [Lancaster and Prells, 2007] can be produced (T_0 and T_n must be nonsingular) by using the following sets to the previous theorem

$$\begin{aligned}
\mathcal{P} &= (0 : k-1), \mathcal{N} = (-n : -k), \\
\mathcal{R}_{\mathcal{P}} &= ((0 : k-2), \ldots, (0 : 0)), \\
\mathcal{R}_{\mathcal{N}} &= ((-n : -k-1), \ldots, (-n : -n)), \\
\mathcal{L}_{\mathcal{P}} &= \varnothing, \mathcal{L}_{\mathcal{N}} = \varnothing.
\end{aligned}$$

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\end{aligned}$$

Example ($L_3(s)$)

Using $k = 3$, $\mathcal{P} = (0, 1, 2)$, $\mathcal{N} = (-5, -4, -3)$, $\mathcal{R}_{\mathcal{P}} = (0, 1, 0)$, $\mathcal{R}_{\mathcal{N}} = (-5, -4, -5)$, $\mathcal{L}_{\mathcal{P}} = \mathcal{L}_{\mathcal{N}} = \varnothing$.

$$L_3(s) = s \begin{bmatrix} 0 & -T_0 & 0 & 0 & 0 \\ -T_0 & -T_1 & 0 & 0 & 0 \\ 0 & 0 & T_3 & T_4 & T_5 \\ 0 & 0 & T_4 & T_5 & 0 \\ 0 & 0 & T_5 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -T_0 & 0 & 0 \\ 0 & -T_0 & -T_1 & 0 & 0 \\ -T_0 & -T_1 & -T_2 & 0 & 0 \\ 0 & 0 & 0 & T_4 & T_5 \\ 0 & 0 & 0 & T_5 & 0 \end{bmatrix}.$$

New symmetric linearizations

Example (Lancaster's modified $L_3(s)$)

Using $k = 3$ and $\mathcal{P} = (1, 2, 0)$, $\mathcal{N} = (-5, -3, -4)$, $\mathcal{R}_{\mathcal{P}} = (1)$, $\mathcal{L}_{\mathcal{N}} = (-4)$, $\mathcal{L}_{\mathcal{P}} = \mathcal{R}_{\mathcal{N}} = \varnothing$

$$L_3^{'}(s) = s \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ I & -T_1 & 0 & 0 & 0 \\ 0 & 0 & T_3 & T_4 & I \\ 0 & 0 & T_4 & T_5 & 0 \\ 0 & 0 & I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & I & 0 & 0 \\ 0 & -T_0 & -T_1 & 0 & 0 \\ I & -T_1 & -T_2 & 0 & 0 \\ 0 & 0 & 0 & T_4 & I \\ 0 & 0 & 0 & I & 0 \end{bmatrix}.$$

Notice that T_0 and T_n can be singular.

Conclusions

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- The members of the new family are operation free.
- Particular members can be constructed according to requirements dictated by the structure of the polynomial matrix.

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