Applications of the full rank factorizations

Rafael Cantó, Beatriz Ricarte y Ana M. Urbano

Institut de Matemàtica Multidisciplinar
Universitat Politècnica de València, Spain
{rcanto, bearibe, amurbano}@mat.upv.es

Outline

$$A \in \mathbb{R}^{n\times m} \quad \operatorname{rank}(A) = r < \min\{n, m\}$$

1. The JORDAN form of $A \in \mathbb{R}^{n\times n}$

   Use the Flanders theorem to obtain:

   - the eigenvalues of A
   - the eigenvectors and the generalized eigenvectors of A

2. The full rank SINGULAR VALUE decomposition of $A \in \mathbb{R}^{n\times m}$

Jordan form of A

THEOREM 1: [CaRiUr, ELA-2009]

$$A \in \mathbb{R}^{n \times r} \quad B \in \mathbb{R}^{r \times n} \quad \operatorname{rank}(A) = \operatorname{rank}(B) = r$$

$$\Downarrow$$

1. $AB \in \mathbb{R}^{n \times n}$ and $BA \in \mathbb{R}^{r \times r}$ have the same elementary divisors with nonzero roots.

2. If $s_1 \geq s_2 \geq \cdots \geq s_p$ (resp. $s'_1 \geq s'_2 \geq \cdots \geq s'_p$) are the size of the Jordan blocks associated with the eigenvalue $\lambda = 0$ of AB (resp. BA), then

$$\boxed{s_i - s'_i = 1} \text{ for all } i$$

Jordan form of A

PROPOSITION 2:

$$A \in \mathbb{R}^{n\times n} \quad \mathrm{rank}(A) = r_1 < n$$

$\boxed{A = F_1 U_1}$ full rank factorization with

- $F_1 \in \mathbb{R}^{n\times r_1} \quad U_1 \in \mathbb{R}^{r_1\times n}$
- $\mathrm{rank}(F_1) = \mathrm{rank}(U_1) = r_1$

Let $\boxed{A_2 = U_1 F_1} \in \mathbb{R}^{r_1\times r_1}$

$$\Downarrow$$

1. A and A_2 have the same elementary divisors with nonzero roots.
2. If $s_1 \geq s_2 \geq \cdots \geq s_p$ (resp. $s'_1 \geq s'_2 \geq \cdots \geq s'_p$) are the size of the Jordan blocks associated with the eigenvalue $\lambda = 0$ of A (resp. A_2), then $s_i - s'_i = 1$, for all i
3. $\mathrm{rank}(A_2^j) = \mathrm{rank}(A^{j+1}), \quad j = 1, 2, \ldots$

Jordan form of A

If $s_1 \geq s_2 \geq \cdots \geq s_q > s_{q+1} = s_{q+2} = \cdots = s_p = 1$

$$J_A = \left[\begin{array}{c|cccc|ccc} J(\lambda) & & & & O & & & \\ \hline & J_{s_1}(0) & O & \ldots & O & O & \ldots & O \\ & O & J_{s_2}(0) & \ldots & O & O & \ldots & O \\ & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ O & O & O & \ldots & J_{s_q}(0) & O & \ldots & O \\ \cline{2-8} & O & O & \ldots & O & J_{s_{q+1}}(0) & \ldots & O \\ & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ & O & O & \ldots & O & O & \ldots & J_{s_p}(0) \end{array}\right]$$

$$\Downarrow$$

$$J_{A_2} = \left[\begin{array}{c|cccc} J(\lambda) & O & O & \ldots & O \\ \hline O & J_{s_1-1}(0) & O & \ldots & O \\ O & O & J_{s_2-1}(0) & \ldots & O \\ \vdots & \vdots & \vdots & & \vdots \\ O & O & O & \ldots & J_{s_q-1}(0) \end{array}\right]$$

Jordan form of A

REMARK 3:

If A_2 is singular

$$\Downarrow$$

Obtain $A_3, \quad \operatorname{rank}(A_3) = \operatorname{rank}(A_2^2) = \operatorname{rank}(A^3)$

Given $A \in \mathbb{R}^{n \times n} \quad \operatorname{rank}(A) = r_1$

$$\Downarrow$$

Construct a sequence of matrices $A_2, A_3, \ldots, A_w, A_{w+1}$ s.t.
$\operatorname{rank}(A_i) = \operatorname{rank}(A^i) = r_i, \ i = 2, 3, \ldots, w+1$ and A_{w+1} nonsingular

- By the Jordan structure of $A_{w+1} \Longrightarrow$ obtain the Jordan structure associated with the nonzero eigenvalues of A
- Using the ranks of $A_2, A_3, \ldots, A_w \Longrightarrow$ construct the Jordan structure associated with the zero eigenvalue of A

Jordan form of A

ALGORITHM 4:

STEP 1. Obtain a full rank factorization of A, i.e.

$$A = F_1 U_1$$
$$r_1 = \text{ number of rows of } U_1$$
$$i = 2$$

STEP 2. Let $U_{i-1}F_{i-1} = A_i$. Obtain a full rank factorization of A_i, i.e.

$$A_i = F_i U_i$$
$$r_i = \text{ number of rows of } U_i$$

STEP 3. If $r_{i-1} > r_i$, then $i = i + 1$, goto Step 2.

Otherwise

Let $w = i - 1$

- $J(\lambda) = \text{jordan}(A_{w+1})$
- $(n - r_1, r_1 - r_2, \ldots, r_{w-1} - r_w)$ the Segre characteristic of A

Obtain $(w, s_2, \ldots, s_{n-r_1})$ the Weyr characteristic of A assoc. with 0

$$\boxed{J_A = \operatorname{diag}(J(\lambda), J_w(0), J_{s_2}(0), \ldots, J_{s_{n-r_1}}(0))}$$

Eigenvectors, $\lambda \neq 0$

Consider the full rank factorization $A = F_1 U_1 \in \mathbb{R}^{n \times n}$

Construct $A_2 = U_1 F_1 \in \mathbb{R}^{r_1 \times r_1}$

Obtain the eigenvectors and the generalized eigenvectors of A associated with its nonzero eigenvalues from the corresponding of A_2

PROPOSITION 5:

If $\{t_r, t_{r-1}, \dots, t_2, t_1\}$ is a Jordan chain of A_2 associated with $\lambda \neq 0$

$$\Downarrow$$

$\{F_1 t_r, F_1 t_{r-1}, \dots, F_1 t_2, F_1 t_1\}$ is a Jordan chain of A associated with $\lambda \neq 0$

Eigenvectors, $\lambda = 0$

Let

$$s_1 \geq s_2 \geq \ldots \geq s_p \geq 1$$

the size of the Jordan blocks associated with the eigenvalue $\lambda = 0$ of A

Consider two cases:

(a) $\operatorname{rank}(A^2) = \operatorname{rank}(A) = r_1$

A_2 nonsingular

$$\Downarrow$$

The eigenvectors of $A = F_1 U_1$ associated with $\lambda = 0$

are the nonzero solutions of the system $U_1 x = 0$

Eigenvectors, $\lambda = 0$

(b) $\text{rank}(A^2) < \text{rank}(A) = r_1$

Suppose that A_2 has q Jordan blocks of size

$s_1 - 1 \geq s_2 - 1 \geq \cdots \geq s_q - 1 \geq 1$ associated with $\lambda = 0$

$$\Downarrow$$

A has, at least, q Jordan blocks of size

$s_1 \geq s_2 \geq \cdots \geq s_q \geq 2$ associated with $\lambda = 0$

PROPOSITION 6:

Let $\{v_{i_{s_i-1}}, v_{i_{s_i-2}}, \ldots, v_{i_2}, v_{i_1}\}$ be a Jordan chain of A_2 of length $s_i - 1$ associated with $\lambda = 0$

$$\Downarrow$$

$\exists \; v_{i_{s_i}} \neq 0$ s.t. $\{v_{i_{s_i}}, F_1 v_{i_{s_i-1}}, F_1 v_{i_{s_i-2}}, \ldots, F_1 v_{i_2}, F_1 v_{i_1}\}$ is a Jordan chain of A of length s_i associated with $\lambda = 0$

Eigenvectors, $\lambda = 0$. Diagram

$$
\begin{array}{llcllc}
 & & & \ker(A^{s_i}) \sim \ker(A^{s_i-1}) & \longrightarrow & v_{i_{s_i}} \\
 & & & & & \downarrow \\
\ker(A_2^{s_i-1}) \sim \ker(A_2^{s_i-2}) & \longrightarrow v_{i_{s_i-1}} & & \ker(A^{s_i-1}) \sim \ker(A^{s_i-2}) & \longrightarrow & F_1 v_{i_{s_i-1}} \\
 & \downarrow & & & & \downarrow \\
\ker(A_2^{s_i-2}) \sim \ker(A_2^{s_i-3}) & \longrightarrow v_{i_{s_i-2}} & & \ker(A^{s_i-2}) \sim \ker(A^{s_i-3}) & \longrightarrow & F_1 v_{i_{s_i-2}} \\
 & \downarrow & & & & \downarrow \\
 & \vdots & \Longrightarrow & & & \vdots \\
 & \downarrow & & & & \downarrow \\
\ker(A_2^2) \sim \ker(A_2) & \longrightarrow \ v_{i_2} & & \ker(A^2) \sim \ker(A) & \longrightarrow & F_1 v_{i_2} \\
 & \downarrow & & & & \downarrow \\
\ker(A_2) & \longrightarrow \ v_{i_1} & & \ker(A) & \longrightarrow & F_1 v_{i_1}
\end{array}
$$

Eigenvectors, $\lambda = 0$. Conclusion

- All Jordan chains of A_2 associated with $\lambda = 0$ can be extended to Jordan chains of A associated with $\lambda = 0$
- If $p = q$, applying Proposition 6 to each chain of A_2 we obtain the corresponding Jordan chain of A
- If $p > q$ besides these Jordan chains we need $p - q$ eigenvectors which are the nonzero solutions of $U_1 x = 0$, linearly independent with the q eigenvectors $\{F_1 v_{1_1}, F_1 v_{2_1}, \ldots, F_1 v_{q_1}\}$ of A

SINGULAR VALUE DECOMPOSITION (SVD)

$$A \in \mathbb{R}^{n\times m}, \quad U^T AV = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p) \quad p = \min\{n, m\}$$

$U \in \mathbb{R}^{n\times n},\ V \in \mathbb{R}^{m\times m}$ orthogonal matrices

Singular values of $A \ \Rightarrow\ \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$

(FRSVD, Full rank SVD)

If $\mathrm{rank}(A) = r < \min\{n, m\}$

$$\Downarrow$$

$$\boxed{A = U_1 S_1 V_1^T}$$

$U_1 \in \mathbb{R}^{n\times r},\ V_1 \in \mathbb{R}^{m\times r}$ with orthonormal columns

$$S_1 = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ nonzero singular values of A

MAIN GOAL: Obtain the FRSVD of A by using its full rank QR factorization

Full rank QR factorization

THEOREM 8:

$$A \in \mathbb{R}^{n\times m} \;\; \mathrm{rank}(A) = r < \min\{n, m\}$$

$$\Downarrow$$

$$\boxed{A = Q_1 R_1}$$

$Q_1 \in \mathbb{R}^{n\times r}$ with orthonormal columns

$R_1 \in \mathbb{R}^{r\times m}$ upper echelon matrix with positive leading entry in each row

R_1 is the upper triangular Cholesky factor of $A^T A$

PROPOSITION 9:

$$A \in \mathbb{R}^{n\times m} \;\; \mathrm{rank}(A) = r < \min\{n, m\}$$

$A = Q_1 R_1$ the full rank QR factorization

$$\Downarrow$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ nonzero singular values of A are

the positive square roots of the eigenvalues of $\boxed{R_1 R_1^T}$

Full rank SVD decomposition

PROPOSITION 10:

$$A \in \mathbb{R}^{n\times m} \;\; \mathrm{rank}(A) = r < \min\{n, m\}$$

$A = Q_1 R_1$ the full rank QR factorization

$\{b_1, b_2, \ldots, b_r\}$ a basis with the orthonormal eigenvectors of $R_1 R_1^T$

associated with the eigenvalues $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2\}$

$$\Downarrow$$

$R_1^T b_1, R_1^T b_2, \ldots, R_1^T b_r$ are orthogonal eigenvectors of $A^T A$ with $\|R_1^T b_i\| = \sigma_i$

From the matrices:

$$\begin{cases} B = [b_1\, b_2\ \ldots, b_r] \\ S_1 = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \\ V_1 = R_1^T B S_1^{-1} \\ U_1 = A V_1 S_1^{-1} \end{cases}$$

we have the following result

Full rank SVD decomposition

THEOREM 11: (FRSVD)

$$A \in \mathbb{R}^{n \times m} \ \ \mathrm{rank}(A) = r < \min\{n, m\}$$

$$\Downarrow$$

$$\boxed{A = U_1 S_1 V_1^T}$$

$U_1 \in \mathbb{R}^{n \times r}, \ V_1 \in \mathbb{R}^{r \times m}$ with orthonormal columns

$$S_1 = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ nonzero singular values of A.

Example-1

Consider $A \in \mathbb{R}^{8 \times 5}$

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 4 \\ 1 & -1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 & 4 \\ 1 & -1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 & 4 \\ 1 & -1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 & 4 \\ 1 & -1 & 0 & 2 & 2 \end{bmatrix}$$

Example-1

The full rank QR factorization of A is

$$A = \begin{bmatrix} \sqrt{2}/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & -\sqrt{2}/4 \\ \sqrt{2}/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & -\sqrt{2}/4 \\ \sqrt{2}/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & -\sqrt{2}/4 \\ \sqrt{2}/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & -\sqrt{2}/4 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 2\sqrt{2} & 2\sqrt{2} & 6\sqrt{2} \\ 0 & 2\sqrt{2} & 2\sqrt{2} & -2\sqrt{2} & 2\sqrt{2} \end{bmatrix} = Q_1 R_1$$

Example-1

Since the eigenvalues and the orthonormal vectors of $R_1 R_1^T$ are

$$R_1 R_1^T = \begin{bmatrix} 96 & 24 \\ 24 & 32 \end{bmatrix} \Rightarrow \begin{cases} \sigma_1^2 = 104, & b_1 = (3/\sqrt{10}, 1/\sqrt{10})^T \\ \sigma_2^2 = 24, & b_2 = (1/\sqrt{10}, -3/\sqrt{10})^T \end{cases}$$

The nonzero singular values of A are $\boxed{\sigma_1 = \sqrt{104}}$ and $\boxed{\sigma_2 = \sqrt{24}}$

From matrices

$$B = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} \quad \text{and} \quad S_1 = \begin{bmatrix} \sqrt{104} & 0 \\ 0 & \sqrt{24} \end{bmatrix}$$

we obtain the full rank SVD decomposition

$$A = U_1 S_1 V_1^T$$

Example-1

$$V_1 = R_1^T B S_1^{-1} \quad = \quad \begin{bmatrix} 3/\sqrt{130} & 1/\sqrt{30} \\ 1/\sqrt{130} & -3/\sqrt{30} \\ 4/\sqrt{130} & -2/\sqrt{30} \\ 2/\sqrt{130} & 4/\sqrt{30} \\ 10/\sqrt{130} & 0 \end{bmatrix}$$

$$U_1 = A V_1 S_1^{-1} \quad = \quad \begin{bmatrix} 1/\sqrt{5} & -1/2\sqrt{5} \\ 1/2\sqrt{5} & 1\sqrt{5} \\ 1/\sqrt{5} & -1/2\sqrt{5} \\ 1/2\sqrt{5} & 1\sqrt{5} \\ 1/\sqrt{5} & -1/2\sqrt{5} \\ 1/2\sqrt{5} & 1\sqrt{5} \\ 1/\sqrt{5} & -1/2\sqrt{5} \\ 1/2\sqrt{5} & 1\sqrt{5} \end{bmatrix}.$$

Example

Consider $A \in \mathbb{R}^{8\times 8}$

$$A = \begin{bmatrix} 2 & 1 & -2 & 1 & -2 & 1 & 2 & 1 \\ 1 & 5 & -3 & -1 & 1 & 1 & 1 & -5 \\ -2 & 1 & 2 & 1 & 2 & 1 & -2 & 1 \\ 3 & -1 & -1 & 5 & -1 & -5 & -1 & 1 \\ -2 & 1 & 2 & 1 & 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & -5 & 1 & 5 & -3 & -1 \\ 2 & 1 & -2 & 1 & -2 & 1 & 2 & 1 \\ -1 & -5 & -1 & 1 & 3 & -1 & -1 & 5 \end{bmatrix}$$

Example

A admits the full rank factorization

$$A = \underbrace{\begin{bmatrix} 2 & 1 & -2 & 1 & -2 & 1 \\ 1 & 5 & -3 & -1 & 1 & 1 \\ -2 & 1 & 2 & 1 & 2 & 1 \\ 3 & -1 & -1 & 5 & -1 & -5 \\ -2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & -5 & 1 & 5 \\ 2 & 1 & -2 & 1 & -2 & 1 \\ -1 & -5 & -1 & 1 & 3 & -1 \end{bmatrix}}_{F_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}}_{U_1}$$

Example

$$\boxed{\begin{array}{r} r_1 = 6 \\ i = 2 \end{array}}$$

$\Longrightarrow$ Construct $A_2 = U_1 F_1$ and the full rank factorization

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 10 & -2 & -2 \\ -4 & 0 & 0 & 4 \\ 2 & -6 & 6 & 2 \\ -4 & 0 & 0 & 4 \\ 0 & -4 & -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} = F_2 U_2$$

$$\operatorname{rank}(A_2) = \operatorname{rank}(A^2) = 4 \quad \Longrightarrow \quad \boxed{r_2 = 4}$$

Example

$\boxed{r_1 > r_2 \;\; \longrightarrow i = 3}$ $\Longrightarrow$ Construct $A_3 = U_2 F_2$ and the full rank factorization

$$A_3 = U_2 F_2 = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 10 & -2 \\ 2 & -2 & 10 \\ -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = F_3 U_3$$

$$\operatorname{rank}(A_3) = \operatorname{rank}(A_2^2) = \operatorname{rank}(A^3) = 3 \quad \Longrightarrow \quad \boxed{r_3 = 3}$$

$\boxed{r_2 > r_3 \;\; \longrightarrow i = 4}$ $\Longrightarrow$ Construct $A_4 = U_3 F_3$ and the full rank factorization

$$A_4 = U_3 F_3 = \begin{bmatrix} 8 & 0 & 0 \\ 2 & 10 & -2 \\ 2 & -2 & 10 \end{bmatrix} = F_4 U_4$$

$$\operatorname{rank}(A_4) = \operatorname{rank}(A_3^2) = \operatorname{rank}(A^4) = 3 \quad \Longrightarrow \quad \boxed{r_4 = 3}$$

$\boxed{r_3 = r_4 \;\; \longrightarrow w = i - 1 = 3}$

Example

The eigenvalues of A_4 are

$$\begin{aligned} \lambda_1 &= 12 \quad \text{with} \quad \text{algeb. mult.} = 1 \\ \lambda_2 &= \ \ 8 \quad \text{with} \quad \text{algeb. mult.} = 2 \end{aligned}$$

and its Jordan structure is $J(\lambda) = \mathrm{diag}(J_1(12), J_2(8))$.

The Segre characteristic of A associated with $\lambda = 0$ is

$$(n - r_1, r_1 - r_2, r_2 - r_3) = (2, 2, 1),$$

and the Weyr characteristic of A associated with $\lambda = 0$ is $(w, s_2) = (3, 2)$.

Then, the Jordan structure of A is $\boxed{J_A = \text{diag}\,(J_1(12),\ J_2(8),\ J_3(0),\ J_2(0))}$

Example

Consider the previous matrix $A \in \mathbb{R}^{8\times 8}$ with $\mathrm{rank}(A) = 6$

$$A = \begin{bmatrix} 2 & 1 & -2 & 1 & -2 & 1 & 2 & 1 \\ 1 & 5 & -3 & -1 & 1 & 1 & 1 & -5 \\ -2 & 1 & 2 & 1 & 2 & 1 & -2 & 1 \\ 3 & -1 & -1 & 5 & -1 & -5 & -1 & 1 \\ -2 & 1 & 2 & 1 & 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & -5 & 1 & 5 & -3 & -1 \\ 2 & 1 & -2 & 1 & -2 & 1 & 2 & 1 \\ -1 & -5 & -1 & 1 & 3 & -1 & -1 & 5 \end{bmatrix}$$

$$J_A = \text{diag}\left(J_1(12),\ J_2(8),\ J_3(0),\ J_2(0)\right).$$

Example

To obtain the Jordan chains of A associated with the nonzero eigenvalues, consider the matrix

$$A_4 = \begin{bmatrix} 8 & 0 & 0 \\ 2 & 10 & -2 \\ 2 & -2 & 10 \end{bmatrix}$$

with Jordan structure $J(\lambda) = \mathrm{diag}(J_1(12),\ J_2(8))$

- For $\lambda_1 = 12$

$$\ker(12I - A_4) = \text{Env}\{t_1 = (0, 1, -1)\}$$

- For $\lambda_2 = 8$

$$\begin{array}{lcc} \ker((8I - A_4)^2) \sim \ker(8I - A_4) & \longrightarrow & t_{2_2} = (1, 0, 0) \\ & & \downarrow \\ \ker(8I - A_4) & \longrightarrow & t_{2_1} = (0, -2, -2) \end{array}$$

Example

- For $\boxed{\lambda_1 = 12}$

$$
\begin{array}{ccc}
\boxed{t_1} & \in & \text{ker}(12I - A_4) \\
\Downarrow & & \\
F_3 t_1 & \in & \text{ker}(12I - A_3) \\
\Downarrow & & \\
F_2 F_3 t_1 & \in & \text{ker}(12I - A_2) \\
\Downarrow & & \\
\boxed{F_1 F_2 F_3 t_1} & \in & \text{ker}(12I - A)
\end{array}
$$

Then

$$
\begin{array}{lcl}
\text{ker}(12I - A_4) = \text{Env}\{t_1 = (0, 1, -1)\} & \rightarrow & \text{ker}(12I - A_3) = \text{Env}\{F_3 t_1\} \rightarrow \\
\text{ker}(12I - A_2) = \text{Env}\{F_2 F_3 t_1\} & \rightarrow & \text{ker}(12I - A) = \text{Env}\{F_1 F_2 F_3 t_1\} \\
& \Downarrow &
\end{array}
$$

$$\text{ker}(12I - A) = \text{Env}\{(0, 1, 0, -1, 0, 1, 0, -1)\}$$

Example

- For $\boxed{\lambda_2 = 8}$

$$\begin{array}{lcc} \ker((8I - A_4)^2) \sim \ker(8I - A_4) & \longrightarrow & \boxed{t_{2_2}} \\ & & \downarrow \\ \ker(8I - A_4) & \longrightarrow & \boxed{t_{2_1}} \end{array}$$

and the corresponding Jordan chain of A is

$$\begin{array}{lcc} \ker((8I - A)^2) \sim \ker(8I - A) & \longrightarrow & \boxed{F_1F_2F_3t_{2_2}} \\ & & \downarrow \\ \ker(8I - A) & \longrightarrow & \boxed{F_1F_2F_3t_{2_1}} \end{array}$$

Example

To obtain the Jordan chains of A associated with $\lambda = 0$, recall that for this eigenvalue:

$$\begin{aligned} A_3 &: \text{ has a Jordan chain of length } 1 \\ A_2 &: \text{ has two Jordan chains of lengths } 2 \text{ and } 1 \\ A &: \text{ has two Jordan chains of lengths } 3 \text{ and } 2. \end{aligned}$$

$$\begin{aligned} A &= F_1 U_1 \\ A_2 &= U_1 F_1 = F_2 U_2 \\ A_3 &= U_2 F_2 = F_3 U_3 \end{aligned}$$

Example

- For $\boxed{\lambda_3 = 0}$

$$
\begin{array}{lccc}
\mathrm{ker}(A_3) & \longrightarrow & z_1 & \\
\hline
\mathrm{ker}(A_2^2) \sim \mathrm{ker}(A_2) & \longrightarrow & r_2 & \\
 & & \downarrow & \\
\mathrm{ker}(A_2) & \longrightarrow & r_1 & s_1 \\
\hline
\mathrm{ker}(A^3) \sim \mathrm{ker}(A^2) & \longrightarrow & w_3 & \\
 & & \downarrow & \\
\mathrm{ker}(A^2) \sim \mathrm{ker}(A) & \longrightarrow & w_2 & u_2 \\
 & & \downarrow & \downarrow \\
\mathrm{ker}(A) & \longrightarrow & w_1 & u_1
\end{array}
$$

Example

z_1 is a nonzero solution of the system

$$A_3 x = F_3 U_3 x = 0 \quad \Longrightarrow \quad U_3 x = 0$$

For instance

$$z_1 = (1, 0, 0, 1)$$

Then

$$r_1 = F_2 z_1 = (0, 0, 0, 4, 0, 4)$$

$$\Downarrow$$

$$\boxed{w_1 = F_1 r_1 = (8, 0, 8, 0, 8, 0, 8, 0)}$$

Example

$$\text{ker}(A_3) \quad \longrightarrow \quad \boxed{z_1}$$

$$\text{ker}(A_2^2) \sim \text{ker}(A_2) \quad \longrightarrow \quad r_2$$

$$\downarrow$$

$$\text{ker}(A_2) \quad \longrightarrow \quad \boxed{r_1 = F_2 z_1} \qquad s_1$$

$$\text{ker}(A^3) \sim \text{ker}(A^2) \quad \longrightarrow \quad w_3$$

$$\downarrow$$

$$\text{ker}(A^2) \sim \text{ker}(A) \quad \longrightarrow \quad w_2 \qquad u_2$$

$$\downarrow \qquad \downarrow$$

$$\text{ker}(A) \quad \longrightarrow \quad \boxed{w_1 = F_1 r_1} \qquad u_1$$

Example

r_2 is a nonzero solution of the system $U_2 x = z_1$. Then,

$$r_2 = (2,0,1,1,1,1) \quad \rightarrow \quad \boxed{w_2 = F_1 r_2 = (2,0,2,4,2,4,2,0)}$$

s_1 is a solution of $U_2 x = 0$, s.t. s_1 and r_1 are l.i.

$$s_1 = (1,0,1,0,0,0) \quad \rightarrow \quad \boxed{u_1 = F_1 s_1 = (0,-2,0,2,0,2,0,-2)}$$

Example

$$\text{ker}(A_3) \longrightarrow \boxed{z_1}$$

$$\begin{array}{lcccc}
\text{ker}(A_2^2) \sim \text{ker}(A_2) & \longrightarrow & \boxed{r_2} & & \\
 & & \downarrow & & \\
\text{ker}(A_2) & \longrightarrow & \boxed{r_1 = F_2 z_1} & & \boxed{s_1}
\end{array}$$

$$\begin{array}{lcccc}
\text{ker}(A^3) \sim \text{ker}(A^2) & \longrightarrow & w_3 & & \\
 & & \downarrow & & \\
\text{ker}(A^2) \sim \text{ker}(A) & \longrightarrow & \boxed{w_2 = F_1 r_2} & & u_2 \\
 & & \downarrow & & \downarrow \\
\text{ker}(A) & \longrightarrow & \boxed{w_1 = F_1 r_1} & & \boxed{u_1 = F_1 s_1}
\end{array}$$

Example

Finally, the vectors w_3 and u_2 are solutions of the systems

$$U_1 x = r_2 \quad \longrightarrow \quad \boxed{v_3 = x = (3, 1, 2, 0, 2, 0, 1, 1)}$$

$$U_1 x = s_1 \quad \longrightarrow \quad \boxed{u_2 = x = (2, 1, 2, -1, 1, -1, 1, 1)}$$

Example

$\text{ker}(A_3) \longrightarrow \boxed{z_1}$

$\text{ker}(A_2^2) \sim \text{ker}(A_2) \longrightarrow \boxed{r_2}$

$\downarrow$

$\text{ker}(A_2) \longrightarrow \boxed{r_1 = F_2 z_1} \quad \boxed{s_1}$

$\text{ker}(A^3) \sim \text{ker}(A^2) \longrightarrow \boxed{w_3}$

$\downarrow$

$\text{ker}(A^2) \sim \text{ker}(A) \longrightarrow \boxed{w_2 = F_1 r_2} \quad \boxed{u_2}$

$\downarrow \quad \downarrow$

$\text{ker}(A) \longrightarrow \boxed{w_1 = F_1 r_1} \quad \boxed{u_1 = F_1 s_1}$