The upper bound for the exponent of a primitive, cone preserving map

R. Loewy

Based on joint work with Bit-Shun Tam

<u>Definition</u> Let $A \in \mathbb{R}^{n,n}$ be a nonnegative matrix (denoted as usual by $A \geq 0$). We say A is <u>primitive</u> if there exists a positive integer l such that

$$A^l > 0.$$

The least such l is called the exponent of A and denoted by

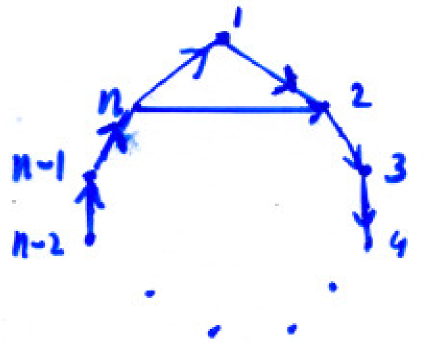
$$\gamma(A).$$

Wielandt's upper bound (1950):

If A is an $n \times n$ primitive matrix then

$$\gamma(A) \le (n-1)^2 + 1 = n^2 - 2n + 2,$$

with equality if and only if the (usual) directed graph associated with A is given by



$\longleftrightarrow$ This is the Wielandt graph

The proof is combinatorial.

Better bound (Neufeld, Shen):

$$\gamma(A) \le (d-1)^2 + 1,$$

where d = degree of the minimum poly. of A.

<u>Our goal</u> obtain Wielandt-type bounds for various related topics.

For us the proper way to consider $A \geq 0$ in $\mathbb{R}^{n,n}$ is as a linear operator that maps the nonnegative orthant $\mathbb{R}^n_+$, where

$$\mathbb{R}_n^+ = \{x \in \mathbb{R}^n : x \geq 0\},$$

into itself. Moreover, A is primitive if and only if $\exists\, l > 0$ such that

A^l maps $\mathbb{R}^n_+ \setminus \{0\}$ into $\operatorname{int} \mathbb{R}^n_+$.

Convex cones

Definition: A proper convex cone is a nonempty subset of $\mathbb{R}^n$ satisfying:

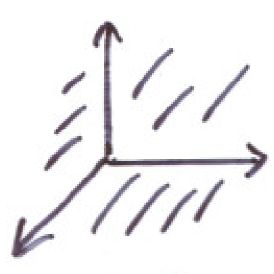
(i) $x, y \in K \Rightarrow x+y \in K$,

(ii) $x \in K, \alpha \geq 0 \Rightarrow \alpha x \in K$,

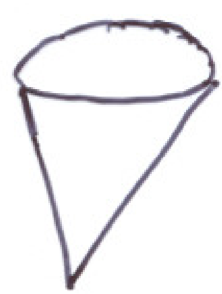
(iii) K is closed,

(iv) $K \cap (-K) = \{0\}$, K is pointed

(v) $\operatorname{int} K \neq \emptyset$. K is full



$\mathbb{R}^3_+$



<u>Definition</u> Let K be a proper cone in $\mathbb{R}^n$ and let $A \in \mathbb{R}^{n,n}$.

(i) We say A is <u>K-nonnegative</u> if $x \in K \Rightarrow Ax \in K$.

(ii) We say A is <u>K-positive</u> if $x \in K \setminus \{0\} \Rightarrow Ax \in \operatorname{int} K$

(iii) We say A is <u>K-primitive</u>, if A is K-nonnegative and A^l is K-positive for some $l > 0$.

(iv) The <u>exponent</u> $\gamma_K(A)$ of a K-primitive matrix A is the smallest l such that A^l is K-positive.

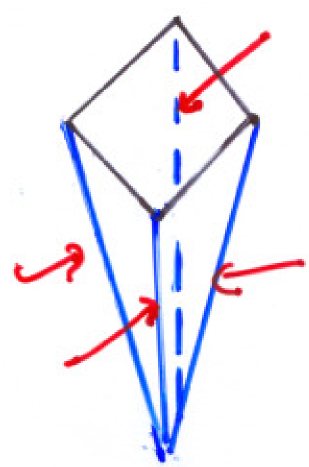
A proper cone in $\mathbb{R}^n$ introduces a partial order $\leq_K$ by

$$x \leq_K y \iff y - x \in K,$$

so

$$K = \{x \in \mathbb{R}^n : x \geq_K 0\}.$$

<u>Definition</u> An <u>extreme ray</u> of K is a one-dimensional face of K.



Notation Let $\mathcal{P}_{m,n}$ denote the set of all proper cones in $\mathbb{R}^n$ having exactly m extreme rays.

A Wielandt type bound for $\gamma_K(A)$

Theorem (L-T)

Let $K \in \mathcal{P}_{m,n}$ and let A be K-primitive. Then,

$$\gamma_K(A) \leq (d-1)(m-1)+1,$$

where $d =$ degree of the minimum poly. of A.

<u>Corollary</u>

Let $K \in \mathcal{P}_{m,n}$ and let A be K-primitive. Then,

$$\gamma_K(A) \leq (n-1)(m-1)+1.$$

A few words about the proof

As in the nonnegative case, a graph $D_K(A)$ (first introduced by Barker and Tam) is used.

The graph has m vertices, namely the extreme rays of K.

Given any extreme ray, we write it as $[x]$, where $0 \neq x \in K$ is any point on that ray.

Given $[x]$ and $[y]$, there is a directed edge from $[x]$ to $[y]$ if and only if

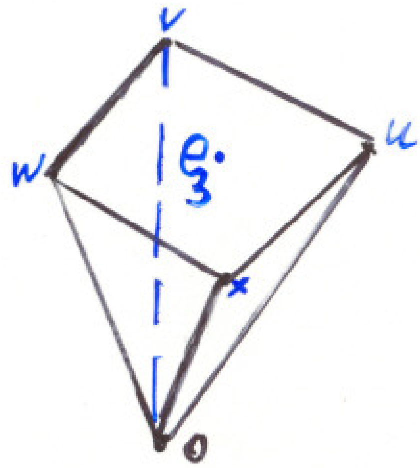
$\exists \alpha > 0$ such that $y \leq_K \alpha A x$.

Example

Let K be the cone generated by

$u=(1,0,1)$; $v=(0,1,1)$; $w=(-1,0,1)$; $x=(0,-1,1)$.

Note that $u+w=v+x$



The extreme rays of K are

$\phi(u)$; $\phi(v)$; $\phi(w)$; $\phi(x)$

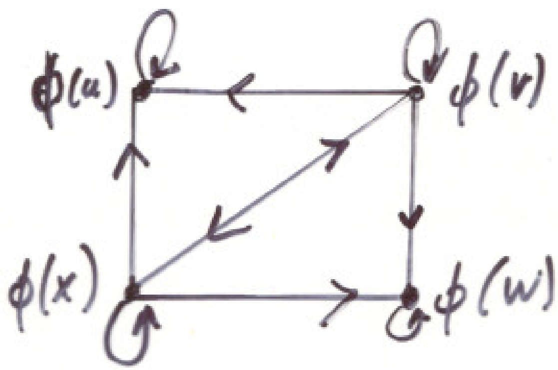
Let $A = \mathrm{diag}(1,0,1)$.

A is an orthogonal projection on $\mathrm{Span}(\{u,w\})$, and

$Au=u$; $Aw=w$; $Av=Ax=e_3=\frac{1}{2}(u+w)=\frac{1}{2}(v+x)$.

So

$D_K(A)$



In case $K = \mathbb{R}^n_+$, $D_K(A)$ reduces to the usual directed graph of A^t.

The proof of the W-type upper bound for $\gamma_K(A)$ analyzes the structure of $D_K(A)$, and in particular its cycles.

There are similarities with the "usual" nonnegative case, but also some <u>significant differences</u>.

For example, $D_K(A)$ does <u>not</u> have to be strongly connected.

Then there is the issue of <u>realizability</u>.

Namely,

given a directed graph on m vertices and $K \in \mathcal{P}_{m,n}$, does there exist a K-primitive matrix A such that $D_K(A)$ is isomorphic to it?

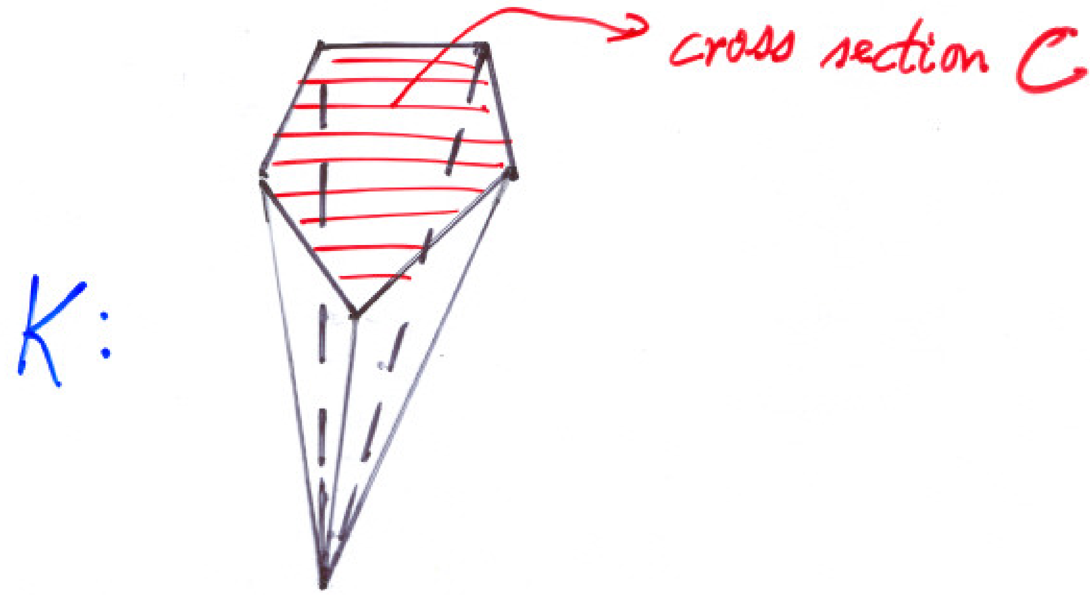
A related quantity

Note that the W-type upper bound for $\gamma_K(A)$ assumes K has finitely many extreme rays.

Such a cone is called polyhedral.

When $\mathcal{C}$ is a polytope containing 0 as an interior point one can define the notion of $\mathcal{C}$-primitive matrix B and for such a matrix define its exponent $\gamma_{\mathcal{C}}(B)$.

There is a natural connection between the notion of $\gamma_K(A)$ (where K is a polyhedral cone in $\mathbb{R}^n$) and $\gamma_C(B)$ (where C is a polytope in $\mathbb{R}^{n-1}$ containing 0 in its interior) and $BC \subseteq C$.



Now suppose that $\|\cdot\|$ is a <u>norm</u> on $\mathbb{R}^n$.
The <u>critical exponent</u> of this norm is defined to be the smallest positive integer l such that

$$1 = \|A^l\| = \|A\| \Rightarrow \|A^j\| = 1 \quad \forall \text{ positive } j.$$

If we denote by $\mathbb{B}$ the unit ball of $\|\cdot\|$ then it is not difficult to see that the critical exponent of $\|\cdot\|$ is equal to

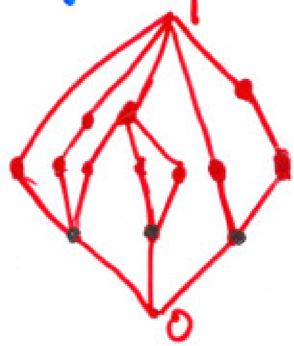
$$\max\{\gamma_{\mathbb{B}}(A) : A \text{ is } \mathbb{B}\text{-primitive}\}$$

Lyubich, Perles, Ptak, . . .

Grinberg's paper

Let X be a finite set which is partially ordered and has least element 0 and largest element 1.

Ex:



Let $\mathcal{M}$ denote the set of all mappings $\varphi: X \to X$ such that

$$\varphi(0) = 0;$$

$$\varphi(1) = 1;$$

$$\varphi(x) \leq \varphi(y) \text{ whenever } x \leq y$$ (monotonicity)

<u>Note</u> $\mathcal{M}$ is a semigroup with identity (w.r.t. composition).

<u>Definition</u> Let M = set of minimal elements in $X \setminus \{0\}$ (atoms), and

$$m = |M|.$$

m is called the <u>width</u> of X

In the ex. on the previous page

$$m = 3.$$

Definition The **height** h of X is the largest positive integer p such that X contains a chain

$$0 < x_1 < x_2 < \cdots < x_p = 1.$$

In the ex. $h = 4$.

Remark We assume that $h \geq 2$.

Definition

(i) Let $\varphi \in \mathcal{M}_0$. We say φ is **primitive** if there exists $k > 0$ such that

$$\varphi^k(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}.$$

(ii) The **least** such k is called the **exponent of φ**, denoted by $\gamma_X(\varphi)$.

Want

Establish W-type upper bound for $\gamma_X(\varphi)$ when $\varphi \in \mathcal{M}$ is primitive.

Discussion

Pick any $x_0 \in M$. Clearly $\varphi(x_0) \neq 0$, so $\exists\, x_1 \in M$ s.t.

$$x_1 \leq \varphi(x_0).$$

Similarly, pick now $x_2 \in M$ s.t.

$$x_2 \leq \varphi(x_1),$$

and continue likewise.

We obtain a sequence

$$x_0, x_1, x_2, x_3, \dots$$

of elements of M satisfying

$$x_{i+1} \leq \varphi(x_i) \quad \forall \quad i = 0, 1, 2, \dots$$

Hence, $\exists \quad 0 \leq p < q \leq m = |M|$ s.t.

$$x_p = x_q,$$

so

$$x_p \leq \varphi^{q-p}(x_p).$$

Applying φ^{q-p} repeatedly we get

$$0 < x_p \leq \varphi^{q-p}(x_p) \leq \varphi^{2(q-p)}(x_p) \leq \dots$$

The def. of height h implies

$$\varphi^{(h-1)(q-p)}(x_p) = \varphi^{h(q-p)}(x_p)$$

Primitivity of φ implies

$$\varphi^{(h-1)(q-p)}(x_p) = \varphi^{h(q-p)}(x_p) = 1.$$

Since

$$x_p \leq \varphi^p(x_0),$$

we conclude

$$\varphi^{(h-1)(q-p)+p}(x_0) = 1. \qquad (*)$$

This leads to

Theorem $\gamma_X(\varphi) \leq hm - m.$

Proof

Consider $(*)$:

$$(h-1)(q-p)+p = (h-2)(q-p)+q \leq (h-2)m+m = hm-m.$$

$\blacksquare$

It is not difficult to show in a similar way that the upper bound in the previous theorem can be improved if X satisfies:

$$\left.\begin{array}{l} \forall\ x \in X \setminus (M \cup \{0\})\ \exists \text{ distinct } y, z \in M : \\ \text{s.t.} \\ \qquad y < x \text{ and } z < x. \end{array}\right\} (**)$$

<u>Theorem</u> Under the additional assumption stated above

$$\gamma_X(\varphi) \leq hm - h - m + 2.$$

How is this related to K-primitive matrices?

So let $K \in \mathcal{P}_{m,n}$ and let A be K-primitive:

Here X will be the lattice $\mathcal{F}$ of all <u>faces</u> of K. It is well known that K has faces of all dimensions, hence

$$h = n.$$

The minimal elements of X are exactly the <u>extreme rays</u>, and we have m of them

A induces a map $\bar{A}$ on the lattice of faces of K as follows:

Given any face F of K $\bar{A}(F)$ is the smallest face of K containing $A(F)$. It is not difficult to see that

$$\gamma_K(A) = \gamma_{\mathcal{F}}(\bar{A}),$$

and that (**) holds for $\mathcal{F}$. Hence

$$\gamma_K(A) \leq nm - n - m + 2 = (n-1)(m-1) + 1.$$

The work of M. Perles

Recently we have learned about the Ph.D. thesis of Perles from 1964, which is relevant to our work. No part of this thesis appeared in ordinary papers. The English summary of the thesis appeared (without proofs) in proceedings of a conference on convexity held in Copenhagen in 1965.

The approach of Perles is lattice theoretic.

He showed (using our terminology) that

if $K \in \mathcal{P}_{m,n}$ and A is K-primitive then

$$\gamma_K(A) \le \begin{cases} (n-1)(m-1) & \text{if } n \text{ is even, and } m \text{ odd} \\ (n-1)(m-1)+1 & \text{otherwise} \end{cases}$$

He also showed that the upper bounds can always be attained.

Our approach yielded a slightly weaker result in the case $n=\text{even}, m=\text{odd}$ and the same result otherwise.

Back to our basic W-type inequality

Recall that $K \in \mathcal{P}_{m,n}$ and A is K-primitive implies

$$\gamma_K(A) \leq (d-1)(m-1)+1.$$

We can also show: If equality holds then

$D_K(A)$ is given by the W-graph;
A is non-derogatory, so $n=d$.

Some further results

Our (L-T) approach yields many additional results. Among them:

Theorem Suppose that $K \in \mathcal{P}_{m,n}$ and A is K-primitive. Suppose also that $D_K(A)$ is strongly connected and let s denote the length of its shortest cycle. Then

$$\gamma_K(A) \leq m + s(d-2).$$

<u>Theorem</u> Suppose that $K \in \mathcal{P}_{m,n}$ and A is K-primitive. Suppose also

$$\gamma_K(A) = (d-1)(m-1).$$

Then, either

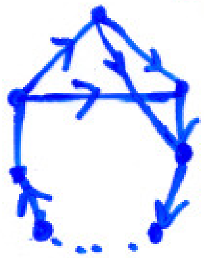
(i) $d = 3$,

or

(ii) $D_K(A)$ is given by the W-graph (and we must have $d = n$)

or

(iii) $D_K(A)$ is given by the graph



(The almost W-graph)

(and we must have $d = n$).

Minimal cones

Any cone in $\mathcal{P}_{n+1,n}$ is called a **minimal cone.**

Here, the max. possible value for $\gamma_K(A)$ is

$$\begin{array}{ll} n^2-n+1 & \text{if } n \text{ is odd,} \\ n^2-n & \text{if } n \text{ is even.} \end{array}$$

We have a full characterization of all minimal cones K and all K-primitive matrices which give this max. value.

3-dimensional cones — $\mathcal{P}_{3,m}$

Here, the max. value for $\gamma_K(A)$ is

$$(3-1)(m-1)+1 = 2m-1.$$

This value is attained exactly for the following one-parameter family of cones K_θ:

$$\theta \in \left(\frac{2\pi}{m}, \frac{2\pi}{m-1}\right),$$

K_θ is generated by the vectors

$$\begin{bmatrix} r_\theta^{j-1}\cos(j-1)\theta \\ r_\theta^{j-1}\sin(j-1)\theta \\ 1 \end{bmatrix}, \quad j=1,2,\ldots,m.$$

Here r_θ is uniquely determined by θ.

Sharpness of the upper bound – general case

As indicated before, in our approach we got the sharpness of the upper bound in all cases of m, n except $n = \text{even}$, $m = \text{odd}$.

In our proof we use among other things the following very nice result of Barnard, Dayawanska, Pearce and Weinberg:

Given a polynomial with nonnegative coefficients one can factor out a pair of complex conjugate roots so that the resulting polynomial also has nonnegative coefficients.