Stability analysis of differential algebraic systems

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Stability analysis for linear differential-algebraic equations DAEs of the form

\[ E(t)\dot{x} = A(t)x + f, \]

with variable coefficients on the half-line \( \mathbb{I} = [0, \infty) \).

They arise as linearization of nonlinear systems

\[ F(t, x, \dot{x}) = 0 \]

around reference solutions.
Applications

- Mechanical multibody systems
- Electrical circuit simulation
- Mechatronic systems
- Chemical reactions
- Semi-discretized PDEs (Stokes, Navier-Stokes)
- Automatically generated coupled systems
Modeling, simulation and software control of automatic gearboxes. Project with Daimler AG (Dissertation: Peter Hamann 2009).
Technological Application

▷ Modeling of multi-physics model: multi-body system, elasticity, hydraulics, friction, . . .

▷ Development of control methods for coupled system.

▷ Real time control of gearbox.

Goal: Decrease full consumption, improve smoothness of switching
Large control system of nonlinear DAEs.
1. Classical theory for ODEs:

2. Extension of theory to DAEs.


Classical spectral theory for ODEs

\[ \dot{x} = f(t, x), \quad t \in \mathbb{I}, \quad x(0) = x^0, \]

with \( x \in C^1(\mathbb{I}, \mathbb{R}^n) \).

By shifting the solution, we may assume that \( x(t) \equiv 0 \).

A constant coefficient system \( \dot{x} = Ax \) with \( A \in \mathbb{R}^{n,n} \) is asymptotically stable if all eigenvalues of \( A \) have negative real part.
Even if for all \( t \in \mathbb{R} \), the matrix \( A(t) \) has all eigenvalues in the left half plane, the system \( \dot{x} = A(t)x \) may be unstable.

**Example** For all \( t \in \mathbb{R} \)

\[
A(t) = \begin{bmatrix}
\cos^2(3t) - 5\sin^2(3t) & -6\cos(3t)\sin(3t) + 3 \\
-6\cos(3t)\sin(3t) + 3 & \sin^2(3t) - 5\cos^2(3t)
\end{bmatrix}
\]

has a double eigenvalue \(-2\) but the solution of \( \dot{x} = A(t)x \),

\[
x(0) = \begin{bmatrix}
c_1 \\
0
\end{bmatrix}
\]

is

\[
x(t) = \begin{bmatrix}
c_1 e^t \cos(3t) \\
-c_1 e^t \cos(3t)
\end{bmatrix}.
\]
Lyapunov exponents

For the linear ODE $\dot{x} = A(t)x$ with bounded coefficient function $A(t)$ and nontrivial solution $x$ we define the *upper and lower Lyapunov exponents*,

$$
\lambda^u(x) = \limsup_{t \to \infty} \frac{1}{t} \ln \|x(t)\|, \quad \lambda^l(x) = \liminf_{t \to \infty} \frac{1}{t} \ln \|x(t)\|.
$$

Since $A$ is bounded, the Lyapunov exponents are finite.

**Theorem (Lyapunov 1907)**

*If the greatest bound of upper Lyapunov exponents for all solutions $\dot{x} = A(t)x$ is negative, then the system is asymptotically stable.*
For the fundamental solution of $\dot{X} = A(t)X$, the Lyapunov exponents for the $i$-th column of $X$ are

$$\lambda^u(Xe_i), \quad \text{and} \quad \lambda^\ell(Xe_i), \quad i = 1, 2, \ldots, n,$$

(1)

where $e_i$ denotes the $i$-th unit vector. W.l.o.g. we assume that the columns of $X$ are ordered such that the upper Lyapunov exponents satisfy

$$\lambda^u(Xe_1) \geq \lambda^u(Xe_2) \geq \ldots \geq \lambda^u(Xe_n).$$
Lyapunov spectrum

**Definition**

Lyapunov 1907, Perron 1930, Daleckii/Krein 1974 The *Lyapunov spectrum* $\Sigma_L$ of $\dot{x} = A(t)x$ is the union of *Lyapunov spectral intervals*

$$\Sigma_L := \bigcup_{i=1}^{n} [\lambda_i^\ell, \lambda_i^u].$$

If each of the Lyapunov spectral intervals shrinks to a point, i.e., if $\lambda_i^\ell = \lambda_i^u \ \forall \ i = 1, 2, \ldots, n$, then the system is called *Lyapunov-regular*. If a system is Lyapunov-regular, then we simply write $\lambda_i$ for the Lyapunov exponents.
Definition

A change of variables $z = T^{-1}x$ with an invertible matrix function $T \in C^1(I, \mathbb{R}^{n \times n})$ is called a **kinematic similarity transformation** if $T$ and $T^{-1}$ are bounded. If $\dot{T}$ is bounded as well, then it is called a **Lyapunov transformation**.

Theorem (Perron 1930)

*For every linear ODE* $\dot{x} = A(t)x$, *there exists a Lyapunov transformation to upper triangular form, and this transformation can be chosen to be pointwise orthogonal.*
Theorem (Lyapunov 1907)

Let $B = [b_{i,j}] \in C(\mathbb{I}, \mathbb{R}^{n \times n})$ be bounded and upper-triangular. Then the system

$$\dot{R} = B(t)R,$$

is Lyapunov-regular if and only if all the limits

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t b_{i,i}(s) ds, \quad i = 1, 2, \ldots, n,$$

exist and these limits coincide with the Lyapunov exponents $\lambda_i, \quad i = 1, 2, \ldots, n.$
Stability of Lyapunov exponents

Definition

Consider a homogeneous linear system \( \dot{x} = A(t)x \) with upper Lyapunov exponents \( \lambda^u_i \) and a perturbed system \( \dot{x} = [A(t) + \Delta A(t)]x \) with upper Lyapunov exponents \( \nu^u_i \), both decreasingly ordered.

- The upper Lyapunov exponents, \( \lambda^u_i \geq \ldots \geq \lambda^u_n \), are called **stable** if for any \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that \( \sup_{t \geq 0} \|\Delta A(t)\| < \delta \) implies

  \[
  |\lambda^u_i - \nu^u_i| < \varepsilon, \quad i = 1, \ldots, n.
  \]

- A fundamental solution matrix \( X \) is called **integrally separated** if for \( i = 1, 2, \ldots, n - 1 \), there exist \( b > 0 \) and \( c > 0 \) such that

  \[
  \frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq ce^{b(t-s)},
  \]

  for all \( t, s \) with \( t \geq s \).
Are Lyapunov exponents stable?

Theorem (see e.g. Dieci/Van Vleck 2006)

i) Integral separation is invariant under Lyapunov transformations (or kinematic similarity transformations).

ii) An integrally separated system has pairwise distinct upper (and pairwise distinct lower) Lyapunov exponents.

iii) Distinct upper Lyapunov exponents are stable if and only if there exists an integrally separated fundamental solution matrix.

iv) Integral separation is a generic property.
Theorem (Dieci/Van Vleck 2002)

A system $\dot{x} = B(t)x$ with $B$ bounded, continuous, and triangular, has an integrally separated fundamental solution matrix iff the diagonal elements of $B$ are integrally separated.

If the diagonal of $B$ is integrally separated, then $\Sigma_L = \Sigma_{CL}$, where

$$\Sigma_{CL} := \bigcup_{i=1}^{n} [\lambda^{\ell}_{i,i}, \lambda^{u}_{i,i}],$$

with

$$\lambda^{\ell}_{i,i} := \liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} b_{i,i}(s) ds, \quad \lambda^{u}_{i,i} := \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} b_{i,i}(s) ds, \quad i = 1, 2, \ldots, n.$$
Definition

Let $x$ be a nontrivial solution of $\dot{x} = A(t)x$. The \textit{(upper) Bohl exponent} $\kappa^u_B(x)$ of this solution is the greatest lower bound of all those numbers $\rho$ for which there exist numbers $N_\rho$ such that

$$\|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\|$$

for any $t \geq s \geq 0$. If such numbers $\rho$ do not exist, then one sets $\kappa^u_B(x) = +\infty$.

Similarly, the \textit{lower Bohl exponent} $\kappa^\ell_B(x)$ is the least lower bound of all those numbers $\rho'$ for which there exist numbers $N'_\rho$ such that

$$\|x(t)\| \geq N'_\rho e^{\rho'(t-s)} \|x(s)\|, \quad 0 \leq s \leq t.$$ 

The interval $[\kappa^\ell_B(x), \kappa^u_B(x)]$ is called the \textit{Bohl interval} of the solution.
Bohl exponents

Theorem (Daleckii/Krein 1974)

Bohl and Lyapunov exponents are related via

$$\kappa_B^\ell(x) \leq \lambda^\ell(x) \leq \lambda^u(x) \leq \kappa_B^u(x).$$

The Bohl exponents are given by

$$\kappa_B^u(x) = \limsup_{s,t\to\infty} \frac{\ln \|x(t)\| - \ln \|x(s)\|}{t - s}, \quad \kappa_B^\ell(x) = \liminf_{s,t\to\infty} \frac{\ln \|x(t)\| - \ln \|x(s)\|}{t - s}.$$

If $A(t)$ is integrally bounded, i.e., if $\sup_{t \geq 0} \int_t^{t+1} \|A(s)\| \, ds < \infty$, then the Bohl exponents are finite.
Bohl exponents characterize the uniform growth rate of solutions, while Lyapunov exponents simply characterize the growth rate of solutions departing from $t = 0$.

If the greatest bound of upper Lyapunov exponents for all solutions $\dot{x} = A(t)x$ is negative, then the system is asymptotically stable. If the same holds for the greatest upper bound of the upper Bohl exponents then the system is (uniformly) exponentially stable.

Bohl exponents are stable without any extra assumption.
Sacker-Sell spectra

Definition

The fundamental matrix solution \( X \) of \( \dot{X} = A(t)X \) is said to admit an exponential dichotomy if there exist a projector \( P : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \) and constants \( \alpha, \beta > 0 \), as well as \( K, L \geq 1 \), such that

\[
\| X(t)PX^{-1}(s) \| \leq Ke^{-\alpha(t-s)}, \quad t \geq s,
\]

\[
\| X(t)(I - P)X^{-1}(s) \| \leq Le^{\beta(t-s)}, \quad t \leq s.
\]

The Sacker-Sell (or exponential-dichotomy) spectrum \( \Sigma_S \) for is given by those values \( \lambda \in \mathbb{R} \) such that the shifted system

\[
\dot{x}_\lambda = [A(t) - \lambda I]x_\lambda
\]

does not have exponential dichotomy. The complement of \( \Sigma_S \) is called the resolvent set.
Theorem (Sacker/Sell 1978)

The property that a system possesses an exponential dichotomy as well as the exponential dichotomy spectrum are preserved under kinematic similarity transformations.

\[ \Sigma_S \text{ is the union of at most } n \text{ disjoint closed intervals, and it is stable.} \]

Furthermore, the Sacker-Sell intervals contain the Lyapunov intervals, i.e.

\[ \Sigma_L \subseteq \Sigma_S. \]
**Theorem (Dieci/Van Vleck 2006)**

Consider $\dot{x} = B(t)x$ with $B$ bounded, continuous, and upper triangular. The Sacker-Sell spectrum of this system and that of the corresponding diagonal system

$$\dot{x} = D(t)x, \quad \text{with } D(t) = \text{diag}(b_{1,1}(t), \ldots, b_{n,n}(t)), \quad t \geq 0,$$

coincide.

Thus, one can retrieve $\Sigma_S$ of $\dot{x} = A(t)x$ from the diagonal elements of the triangularized system.
Stability Theory for DAEs

- Lyapunov theory for regular constant coeff. DAEs Stykel 2002
- Index 1 systems Ascher/Petzold 1993,
- Systems of tractability index $\leq 2$, Tischendorf 1994, Hanke/Macana/März 1998,
- Exponential dichotomy in bound. val. problems, Lentini/März 1990.
- General theory for linear DAEs Linh/M. 2008
- Perturbation theory Linh/M./Van Vleck 2009
We put $E(t)\dot{x} = A(t)x + f(t)$ and its derivatives up to order $\mu$ into a large DAE

$$M_k(t)\dot{z}_k = N_k(t)z_k + g_k(t), \quad k \in \mathbb{N}_0$$

for $z_k = (x, \dot{x}, \ldots, x^{(k)})$.

$$M_2 = \begin{bmatrix} E & 0 & 0 \\ A - \dot{E} & E & 0 \\ \dot{A} - 2\ddot{E} & A - \dot{E} & E \end{bmatrix}, \quad N_2 = \begin{bmatrix} A & 0 & 0 \\ \dot{A} & 0 & 0 \\ \ddot{A} & 0 & 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} x \\ \dot{x} \\ \dddot{x} \end{bmatrix}. $$
Theorem (Kunkel/M. 1996)

Under some constant rank assumptions, for a square regular linear DAE there exist integers $\mu$, $a$, $d$ such that:

1. $\text{rank } M_\mu(t) = (\mu + 1)n - a$ on $\mathbb{I}$, and there exists a smooth matrix function $Z_2$ with $Z_2^T M_\mu(t) = 0$.

2. The first block column $\hat{A}_2$ of $Z_2^* N_\mu(t)$ has full rank $a$ so that there exists a smooth matrix function $T_2$ such that $\hat{A}_2 T_2 = 0$.

3. $\text{rank } E(t) T_2 = d = n - a$ and there exists a smooth matrix function $Z_1$ of size $(n, d)$ with $\text{rank } \hat{E}_1 = d$, where $\hat{E}_1 = Z_1^T E$. 
The quantity $\mu$ is called the strangeness-index. It describes the smoothness requirements for the inhomogeneity.

It generalizes the usual differentiation index to general DAEs (and counts slightly differently).

We obtain a numerically computable strangeness-free formulation of the equation with the same solution.

$$
\dot{E}_1(t)x = \hat{A}_1(t)x + \hat{f}_1(t), \quad d \text{ differential equations}
$$

$$
0 = \hat{A}_2(t)x + \hat{f}_2(t), \quad a \text{ algebraic equations}
$$

where $\hat{A}_1 = Z_1^T A$, $\hat{f}_1 = Z_1^T f$, and $\hat{f}_2 = Z_2^T g_\mu$.

The reduced system is strangeness-free. This is a Remodelling! For the theory we may assume that this has been done.
**Definition**

A matrix function $X \in C^1(\mathbb{I}, \mathbb{R}^{n \times k})$, $d \leq k \leq n$, is called **fundamental solution matrix of** $E(t) \dot{X} = A(t)X$ if each of its columns is a solution to $E(t)\dot{x} = A(t)x$ and $\text{rank } X(t) = d$, for all $t \geq 0$.

A fundamental solution matrix is said to be **maximal** if $k = n$ and **minimal** if $k = d$, respectively.

A maximal fundamental matrix solution, denoted by $X(t, s)$, is called **normalized** if it satisfies the *projected initial condition*

$$E(0)(X(0, 0) - I) = 0.$$  

Every fundamental solution matrix has exactly $d$ linearly independent columns and a minimal fundamental matrix solution can be easily made maximal by adding $n - d$ zero columns.
Lyapunov exponents for DAEs

Definition

For a fundamental solution matrix $X$ of a strangeness-free DAE system $E(t)\dot{x} = A(t)x$, and for $d \leq k \leq n$, we introduce

$$\lambda^u_i = \limsup_{t \to \infty} \frac{1}{t} \ln \|X(t)e_i\| \quad \text{and} \quad \lambda^\ell_i = \liminf_{t \to \infty} \frac{1}{t} \ln \|X(t)e_i\|, \quad i = 1, 2, \ldots, k.$$ 

The $\lambda^u_i, i = 1, 2, \ldots, n$, are called (upper) **Lyapunov exponents** and the intervals $[\lambda^\ell_i, \lambda^u_i], i = 1, 2, \ldots, d$, are called **Lyapunov spectral intervals**. The DAE system is said to be **Lyapunov-regular** if $\lambda^\ell_i = \lambda^u_i, i = 1, 2, \ldots, d$. 

Stability analysis for DAEs 28 / 58
Definition

Suppose that \( W \in C(I, \mathbb{R}^{n \times n}) \) and \( T \in C^1(I, \mathbb{R}^{n \times n}) \) are pointwise nonsingular matrix functions such that \( T \) and \( T^{-1} \) are bounded. Then the transformed DAE system

\[
\tilde{E}(t)\dot{x} = \tilde{A}(t)x,
\]

with \( \tilde{E} = WET \), \( \tilde{A} = WAT - WET \dot{T} \) and \( x = T\tilde{x} \) is called \textit{globally kinematically equivalent} to \( E(t)\dot{x} = A(t)x \). If, furthermore, also \( W \) and \( W^{-1} \) are bounded then we call this a \textit{strong global kinematical equivalence transformation}. 
Lemma

Consider a strangeness-free DAE \( E(t) \dot{x} = A(t)x \) with continuous coefficients and a minimal fundamental solution matrix \( X \). Then there exist pointwise orthogonal matrix functions \( U \in C(\mathbb{I}, \mathbb{R}^{n \times n}) \) and \( V \in C^1(\mathbb{I}, \mathbb{R}^{n \times n}) \) such that in \( E \dot{X} = AX \) the change of variables \( X = VR \), with \( R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \) and \( R_1 \in C^1(\mathbb{I}, \mathbb{R}^{d \times d}) \) and the multiplication from the left with \( U^T \) leads to the system

\[
E_d \dot{R}_1 = A_d R_1,
\]

where \( E_d := U_1^T EV_1 \) is pointwise nonsingular and \( A_d := U_1^T AV_1 - U_1^T E \dot{V}_1 \). Here \( U_1, V_1 \) are the matrix functions consisting of the first \( d \) columns of \( U, V \), respectively.
Theorem

Consider a reduced strangeness-free DAE system. If the coefficient matrices are sufficiently smooth, then there exists an orthogonal matrix function $\hat{Q} \in C^1(\mathbb{I}, \mathbb{R}^{n \times n})$ such that with $\hat{x} = \hat{Q}^T x$, the submatrix $E_1$ is compressed, i.e., the transformed system has the form

$$
\begin{bmatrix}
\hat{E}_{11} & 0 \\
0 & 0
\end{bmatrix}
\dot{\hat{x}} =
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix}
\hat{x}, \quad t \geq 0,
$$

Furthermore, this system is still strangeness-free and thus $\hat{E}_{11}$ and $\hat{A}_{22}$ are pointwise nonsingular.

The associated underlying (implicit) ODE is,

$$
\hat{E}_{11}\dot{\hat{x}}_1 = \hat{A}_s \hat{x}_1,
$$

where $\hat{A}_s := \hat{A}_{11} - \hat{A}_{12} \hat{A}_{22}^{-1} \hat{A}_{21}$.
Theorem

Let $\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21})$ be the upper Lyapunov exponent of the matrix function $\hat{A}_{22}^{-1}\hat{A}_{21}$. If the boundedness condition

$$\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21}) \leq 0$$

holds, then the upper Lyapunov exponents of the transformed DAE and its underlying ODE coincide if they are both ordered decreasingly.
Consider perturbed DAEs

\[(E(t) + \Delta E(t)) \dot{x} = (A(t) + \Delta A(t))x, \quad t \geq 0,\]

with special perturbations \((\Delta E, \Delta A), \quad \Delta E, \Delta A \in C(I, \mathbb{R}^{n \times n})\) that are sufficiently smooth and small enough such that by appropriate orthogonal transformation we obtain

\[
\begin{bmatrix}
\hat{E}_{11} + \Delta \hat{E}_{11} & 0 \\
0 & 0
\end{bmatrix}
\dot{x} =
\begin{bmatrix}
\hat{A}_{11} + \Delta \hat{A}_{11} & \hat{A}_{12} + \Delta \hat{A}_{12} \\
\hat{A}_{21} + \Delta \hat{A}_{21} & \hat{A}_{22} + \Delta \hat{A}_{22}
\end{bmatrix} x, \quad t \geq 0.
\]

If this is the case then we say that the perturbations are \textit{admissible}.
Definition

The upper Lyapunov exponents $\lambda^u_1 \geq \ldots \geq \lambda^u_d$ are said to be stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that the conditions
\[
\sup_t \left\| \Delta \tilde{E}(t) \right\| < \delta, \quad \sup_t \left\| \Delta \tilde{A}(t) \right\| < \delta
\]
on admissible perturbations imply that the perturbed DAE system is strangeness-free and
\[
|\lambda^u_i - \gamma^u_i| < \epsilon, \quad \forall i = 1, 2, \ldots, d,
\]
where the $\gamma^u_i$ are the ordered upper Lyapunov exponents of the perturbed system.

A DAE system and an admissibly perturbed system are called asymptotically equivalent if they are strangeness-free and
\[
\lim_{t \to \infty} \| \Delta E(t) \| = \lim_{t \to \infty} \| \Delta A(t) \| = 0.
\]
Theorem

Suppose that the DAE and an admissibly perturbed DAE are asymptotically equivalent. If the Lyapunov exponents are stable then $\lambda_i^u = \gamma_i^u$, for all $i = 1, 2, \ldots, d$, where again the $\gamma_i^u$ are the ordered upper Lyapunov exponents of the perturbed system.
Integral separation

**Definition**

A minimal fundamental solution matrix $X$ for a strangeness-free DAE is called *integrally separated* if for $i = 1, 2, \ldots, d - 1$ there exist $b > 0$ and $c > 0$ such that

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq ce^{b(t-s)},$$

for all $t, s$ with $t \geq s \geq 0$. 
Proposition

Consider a strangeness-free DAE.

1. If the DAE is integrally separated then the same holds for any strongly globally kinematically equivalent system.

2. If the DAE is integrally separated, then it has pairwise distinct upper and pairwise distinct lower Lyapunov exponents.

3. If $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded, then the DAE is integrally separated if and only if and the underlying ODE is integrally separated.
Stability of exponents

Theorem

Consider a strangeness-free DAE and its transformed system, satisfying that

\[
\hat{A}_{22}^{-1} \hat{A}_{21}, \; \hat{A}_{12} \hat{A}_{22}^{-1}, \; \hat{E}_{11}, \; \hat{E}_{11}^{-1} (\hat{A}_{11} - \hat{A}_{12} \hat{A}_{22}^{-1} \hat{A}_{21}),
\]

are bounded. The DAE has \( d \) pairwise distinct upper and pairwise distinct lower Lyapunov exponents and they are stable iff it is integrally separated.

If \( \hat{A}_{22}^{-1} \hat{A}_{21}, \; \hat{A}_{12} \hat{A}_{22}^{-1}, \; \hat{E}_{11}, \; \text{and} \; \hat{E}_{11}^{-1} \) are bounded, then the system has an integrally separated fundamental solution matrix iff its adjoint has.
Example For the DAE system

\[
\begin{align*}
\dot{x}_1 &= x_1 + x_2, & 0 &= x_1 - x_3, \\
\dot{x}_2 &= x_2, & 0 &= x_2 - e^{-t}x_4,
\end{align*}
\]

the underlying ODE is not integrally separated, but the Lyapunov exponents are stable and the minimal fundamental solution

\[
X(t) = \begin{bmatrix}
e^t & te^t \\
0 & e^t \\
e^t & te^t \\
0 & e^{2t}
\end{bmatrix},
\]

is integrally separated. However, the Lyapunov exponents of the DAE (\{1, 2\}), are not stable.
**Definition**

Consider a strangeness-free DAE. For a scalar $\lambda \in \mathbb{R}$, the DAE system

$$E(t)\dot{x} = [A(t) - \lambda E(t)]x, \quad t \geq 0,$$

is called a *shifted DAE system*. The shifted DAE transforms as

$$
\begin{bmatrix}
\hat{E}_{11} & 0 \\
0 & 0
\end{bmatrix}
\hat{x} =
\begin{bmatrix}
\hat{A}_{11} - \lambda \hat{E}_{11} & \hat{A}_{12} \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix}
\hat{x}, \quad t \geq 0,
$$

and clearly, the shifted DAE system inherits the strangeness-free property from the original DAE.
Definition

A strangeness-free DAE system is said to have an exponential dichotomy if for a maximal fundamental solution $\hat{X}(t)$, there exists a projection matrix $P \in \mathbb{R}^{d \times d}$ and constants $\alpha, \beta > 0$, and $K, L \geq 1$ such that

$$\left\| \hat{X}(t) \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^-(s) \right\| \leq Ke^{-\alpha(t-s)}, \quad t \geq s,$$

$$\left\| \hat{X}(t) \begin{bmatrix} I_d - P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^-(s) \right\| \leq Le^{\beta(t-s)}, \quad t \leq s.$$ 

Here $X^-(s)$ is a reflexive generalized inverse.

Theorem

A strangeness-free DAE system has an exponential dichotomy if and only if in the transformed system $\hat{A}_{22}^{-1} \hat{A}_{21}$ is bounded and the underlying ODE has an exponential dichotomy.
Sacker-Sell spectra

Definition

- The **Sacker-Sell (or exponential dichotomy) spectrum** of a strangeness-free DAE system is defined by
  \[ \Sigma_S := \{ \lambda \in \mathbb{R}, \text{ the shifted DAE does not have an exponential dichotomy} \}. \]

- The complement of \( \Sigma_S \) is called the **resolvent set**.

- The Sacker-Sell spectrum of a DAE system does not depend on the choice of the orthogonal change of basis that brings it to the transformed system.
Theorem

Consider a DAE and suppose that in the transformed DAE \( \hat{A}_2^{-1} \hat{A}_{21} \) is bounded.

- The Sacker-Sell spectrum is exactly the Sacker-Sell spectrum of the underlying ODE. It consists of at most \( d \) closed intervals.
- If the Sacker-Sell spectrum of the DAE system is given by \( d \) disjoint closed intervals, then there exists a minimal fundamental solution matrix \( \hat{X} \) with integrally separated columns.
- In this case it is given exactly by the \( d \) (not necessarily disjoint) Bohl intervals associated with the columns of \( \hat{X} \) and \( \Sigma_L \subseteq \Sigma_S \).
Theorem

Consider a strangeness-free DAE system
\[
\hat{A}_{22}^{-1}\hat{A}_{21}, \hat{A}_{12}\hat{A}_{22}^{-1}, \hat{E}_{11}, \hat{E}_{11}^{-1}(\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}),
\]
are bounded and for \( k \leq d \), \( \Sigma_S = \bigcup_{i=1}^{k}[a_i, b_i] \).

Consider an admissible perturbation. Let \( \varepsilon > 0 \) be sufficiently small such that \( b_i + \varepsilon < a_{i+1} - \varepsilon \) for some \( i \), \( 0 \leq i \leq k \). For \( i = 0 \) and \( i = k \), set \( b_0 = -\infty \) and \( a_{k+1} = \infty \), respectively. Then, there exists \( \delta > 0 \) such that the inequality
\[
\max\{\sup_t \|\Delta E(t)\|, \sup_t \|\Delta A(t)\|\} \leq \delta,
\]
implies that the interval \((b_i + \varepsilon, a_{i+1} - \varepsilon)\) is contained in the resolvent of the perturbed DAE system.
Corollary

Let the assumptions of the last Theorem hold and let \( \varepsilon > 0 \) be sufficiently small such that
\[
b_{i-1} + \varepsilon < a_i - \varepsilon < a_i \leq b_i < b_i + \varepsilon < a_{i+1} - \varepsilon, \text{ for } 0 \leq i \leq k.
\]
For \( i = 0 \) and \( i = k \), set \( b_0 = -\infty \) and \( a_{k+1} = \infty \), respectively. Then, there exists \( \delta > 0 \) such that the inequality
\[
\max\{\sup_t \|\Delta \tilde{E}(t)\|, \sup_t \|\Delta \tilde{A}(t)\|\} \leq \delta,
\]
implies that the Sacker-Sell interval \([a_i, b_i]\) either remains a Sacker-Sell interval under the perturbation or it is possibly split into several new intervals, but the smallest left end-point and the largest right end-point stay in the interval \([a_i - \varepsilon, b_i + \varepsilon]\).
For the numerical computation we have first have to obtain the strangeness-free form of $E(t), A(t)$. It can be obtained pointwise for every $t$ via the FORTRAN code GELDA Kunkel/M./Rath/Weickert 1997 or the corresponding MATLAB version Kunkel/M./Seidel 2005. It comes in the form

$$
\begin{bmatrix}
E_1(t) \\
0
\end{bmatrix} \dot{x} = \begin{bmatrix}
A_1(t) \\
A_2(t)
\end{bmatrix} x
$$

where $A_2$ is full row rank.
Suppose that the original DAE has sufficiently smooth coefficients.

There exists a pointwise nonsingular, upper triangular matrix function $\tilde{A}_{22} \in C^1(\mathbb{I}, \mathbb{R}^{(n-d) \times (n-d)})$ and a pointwise orthogonal matrix function $\tilde{Q} \in C^1(\mathbb{I}, \mathbb{R}^{n \times n})$ such that

$$A_2 = \begin{bmatrix} 0 & \tilde{A}_{22} \end{bmatrix} \tilde{Q}.$$ 

To make the factorization unique and to obtain smoothness, we require the diagonal elements of $\tilde{A}_{22}$ to be positive.

Alternatively we can derive differential equations for $\tilde{Q}$ (or its Householder factors) and to solve the corresponding initial value problems.
The transformation $\tilde{x} = \tilde{Q}^T x$ leads to

$$
\begin{bmatrix}
\tilde{E}_{11} & \tilde{E}_{12} \\
0 & 0
\end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \tilde{Q}, \quad
\begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} \\
0 & \tilde{A}_{22}
\end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tilde{Q} - \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{\tilde{Q}}.
$$

To evaluate $\dot{\tilde{Q}}$ at time $t$, we may use either a finite difference formula or the smooth QR method derived in Kunkel/M. 1991. The solution component $\tilde{x}_2$ associated with the algebraic equations is simply 0, thus we only have to deal with $\tilde{x}_1$. 
Basic Idea for Computing Exponents

- Determine for every point $t$
  
  $\mathcal{E} = [e_{ij}] = P^T \tilde{E}_{11} Q$, $A = [a_{ij}] = P^T \tilde{A}_{11} Q - P^T \tilde{E}_{11} \dot{Q} = P^T \tilde{A}_{11} Q - \mathcal{E} Q^T \dot{Q}$

  such that $\mathcal{E}$, $A$ are upper triangular.

- Determine strictly lower triangular part of the skew symmetric $S(Q) = Q^T \dot{Q}$ by corresponding part of $\mathcal{E}^{-1} P^T \tilde{A}_{11} Q$ and the remaining part by skew-symmetry.

- Determine $P$ and $\mathcal{E}$ via a smooth QR-factorization $\tilde{E}_{11} Q = P \mathcal{E}$.

- Keep orthogonality via orthogonal integrators Hairer/Lubich/Wanner 2002 or projected ODE integrators Dieci/Van Vleck 2003.

- Compute the spectral intervals from
  
  $\mathcal{E}_1(t) \dot{R}_1 = A_1(t) R_1$, \quad $t \in \mathbb{I}$,

  where $R_1$ is the fundamental solution matrix of the triangularized underlying implicit ODE.
Compute a smooth QR factorization of $A_2$

$$
\begin{bmatrix}
E_1 & U_1^T \tilde{E}_{12} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{z} \\
\ddot{z}
\end{bmatrix}
=
\begin{bmatrix}
A_1 & U_1^T \tilde{A}_{12} \\
0 & \tilde{A}_{22}
\end{bmatrix}
\begin{bmatrix}
z \\
\ddot{z}
\end{bmatrix},
$$

with $E_1$, $A_1$, $\tilde{A}_{22}$ upper triangular.

Apply ODE methods of Dieci/Van Vleck to

$E_1(t) \dot{R}_1 = A_1(t)R_1$, $t \in \mathbb{I}$,

Compute

$$
\lambda_i(t_j) = \frac{1}{t_j} \ln[R_1(t_j)]_{i,i} = \frac{1}{t_j} \ln \prod_{\ell=1}^{j} [\Theta_{\ell}]_{i,i} = \frac{1}{t_j} \sum_{\ell=1}^{j} \ln[\Theta_{\ell}]_{i,i}, \quad i = 1, 2, \ldots, d.
$$

Solve optimization problems $\inf_{\tau \leq t \leq T} \lambda_i(t)$ and $\sup_{\tau \leq t \leq T} \lambda_i(t)$, $i = 1, 2, \ldots, d$ for a given $\tau \in (0, T)$. 

Stability analysis for DAEs 50 / 58
Discrete QR algorithm

- Take a mesh \(0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T\).

- Compute the fundamental solution \(X[j]\) on \([t_{j-1}, t_j]\) by solving
  \[
  E \dot{X} = AX, \quad t_{j-1} \leq t \leq t_j, \quad X(t_{j-1}) = \chi_{j-1},
  \]
  using the DAE integrator \texttt{GELDA}.

- Determine QR factorizations
  \[
  \tilde{Q}(t_j)^T X[j](t_j) = \begin{bmatrix} Z_j \\ 0 \end{bmatrix}, \quad Z_j = Q_j \Theta_j, \quad j = 1, 2, \ldots, N
  \]

- Compute
  \[
  \lambda_i(t_j) = \frac{1}{t_j} \ln[R_1(t_j)]_{i,i} = \frac{1}{t_j} \ln \prod_{\ell=1}^j [\Theta_{\ell}]_{i,i} = \frac{1}{t_j} \sum_{\ell=1}^j \ln[\Theta_{\ell}]_{i,i}, \quad i = 1, 2, \ldots, d
  \]

- Solve the optimization problems
  \[
  \inf_{t_j \leq t \leq T} \lambda_i(t) \quad \text{and} \quad \sup_{t_j \leq t \leq T} \lambda_i(t), \quad i = 1, 2, \ldots, d
  \]
  for a given \(\tau \in (0, T)\).
Lyapunov regular $2 \times 2$ DAE $\lambda_1 = 5, \lambda_2 = 0$. 

<table>
<thead>
<tr>
<th>$T$</th>
<th>$h$</th>
<th>$\lambda_1$</th>
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<th>CPU(s)</th>
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Lyapunov regular $2 \times 2$ DAE $\lambda_1 = 5, \lambda_2 = 0.$

<table>
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<tr>
<th>$T$</th>
<th>$h$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>CPU(s)</th>
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Lyapunov exp. via cont. QR-Euler method

Lyapunov non regular $2 \times 2$ DAE with Lyapunov intervals $[-1, 1]$, $[-6, -4]$.

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Sacker-Sell interv. via cont. QR-Euler

Sacker-Sell intervals $[-\sqrt{2}, \sqrt{2}]$ and $[-5 - \sqrt{2}, -5 + \sqrt{2}]$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$H$</th>
<th>$h$</th>
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<th>$[\kappa^l_2, \kappa^u_2]$</th>
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</tbody>
</table>
The classical Theory of Lyapunov/Bohl/Sacker-Sell has been extended to linear DAEs with variable coefficients.

Boundedness conditions and strangeness-free formulations are the key tools.

In principle we can compute spectral intervals.

Numerical methods for computing the Sacker-Sell spectra of DAEs extending work of Dieci/Van Vleck. They are expensive.

Perturbation theory of Dieci/Van Vleck for ODEs has been extended and the methods have been modified to deal only with partial exponents (e.g. the largest Sacker-Sell interval) Linh/M./Van Vleck Dec. 2009

SVD based methods for more accurate spectra.
Future work

- Extension to analysis of nonlinear systems
- Production code for classes special classes of DAEs
- Extension to operator DAEs/PDEs
Thank you very much for your attention.