Stability analysis of differential algebraic systems

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Mathematics for key technologies

- Stability analysis for linear differential-algebraic equations DAEs of the form

$$E(t)\dot{x} = A(t)x + f,$$

with variable coefficients on the half-line $\mathbb{I} = [0, \infty)$.

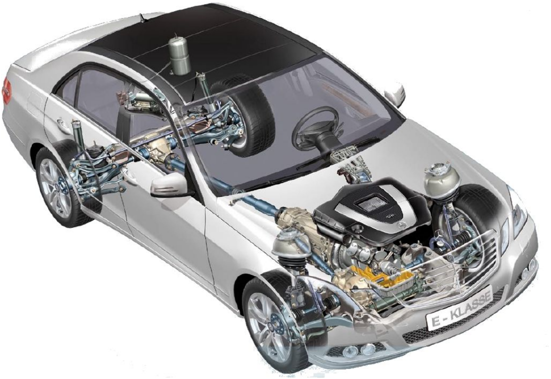
- They arise as linearization of nonlinear systems

$$F(t, x, \dot{x}) = 0$$

around reference solutions.

- Mechanical multibody systems
- Electrical circuit simulation
- Mechatronic systems
- Chemical reactions
- Semi-discretized PDEs (Stokes, Navier-Stokes)
- Automatically generated coupled systems

Modeling, simulation and software control of automatic gearboxes. Project with Daimler AG (Dissertation: Peter Hamann 2009).



Technological Application

- Modeling of multi-physics model: multi-body system, elasticity, hydraulics, friction, ...
- Development of control methods for coupled system.
- Real time control of gearbox.

Goal: Decrease full consumption, improve smoothness of switching
Large control system of nonlinear DAEs.

1. Classical theory for ODEs:
2. Extension of theory to DAEs.
3. Numerical computation of spectral intervals for DAEs (Extending work of Dieci/Van Vleck 1995-2007 for ODEs).

References: V.H. Linh and V.M.. *Lyapunov, Bohl and Sacker-Sell Spectral Intervals for Differential-Algebraic Equations.* J. DYNAMICS AND DIFF. EQUATIONS, Vol. 21, 153–194, 2009
V.H. Linh and V.M. and E. Van Vleck. *QR* Methods and Error Analysis for Computing Lyapunov and Sacker-Sell Spectral Intervals for Linear Differential-Algebraic Equations. To appear in ADV. COMP. MATH., 2010

Spectral theory for ODEs

Classical spectral theory for ODEs

$$\dot{x} = f(t, x), \quad t \in \mathbb{I},\ x(0) = x^0,$$

with $x \in C^1(\mathbb{I}, \mathbb{R}^n)$.
By shifting the solution, we may assume that $x(t) \equiv 0$.
A constant coefficient system $\dot{x} = Ax$ with $A \in \mathbb{R}^{n,n}$ is asymptotically stable if all eigenvalues of A have negative real part.

Asympt. stab. of var. coefficient ODEs

Even if for all $t \in \mathbb{R}$, the matrix $A(t)$ has all eigenvalues in the left half plane, the system $\dot{x} = A(t)x$ may be unstable.

Example For all $t \in \mathbb{R}$

$$A(t) = \begin{bmatrix} \cos^2(3t) - 5\sin^2(3t) & -6\cos(3t)\sin(3t) + 3 \\ -6\cos(3t)sin(3t) + 3 & \sin^2(3t) - 5\cos^2(3t) \end{bmatrix}$$

has a double eigenvalue -2 but the solution of $\dot{x} = A(t)x$, $x(0) = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$ is

$$x(t) = \begin{bmatrix} c_1 e^t \cos(3t) \\ -c_1 e^t \cos(3t) \end{bmatrix}.$$

Lyapunov exponents

For the linear ODE $\dot{x} = A(t)x$ with bounded coefficient function $A(t)$ and nontrivial solution x we define the *upper and lower Lyapunov exponents*,

$$\lambda^u(x) = \limsup_{t\to\infty} \frac{1}{t} \ln \|x(t)\|\,, \quad \lambda^l(x) = \liminf_{t\to\infty} \frac{1}{t} \ln \|x(t)\|\,.$$

Since A is bounded, the Lyapunov exponents are finite.

Theorem (Lyapunov 1907)

If the greatest bound of upper Lyapunov exponents for all solutions $\dot{x} = A(t)x$ is negative, then the system is asymptotically stable.

Lyapunov exp. of fundamental sol'n

For the fundamental solution of $\dot{X} = A(t)X$, the Lyapunov exponents for the i-th column of X are

$$\lambda^u(Xe_i), \quad \text{and} \quad \lambda^\ell(Xe_i), \quad i = 1, 2, ..., n, \tag{1}$$

where e_i denotes the i-th unit vector. W.l.o.g. we assume that the columns of X are ordered such that the upper Lyapunov exponents satisfy

$$\lambda^u(Xe_1) \geq \lambda^u(Xe_2) \geq ... \geq \lambda^u(Xe_n).$$

Definition

Lyapunov 1907, Perron 1930, Daleckii/Krein 1974 The *Lyapunov spectrum* Σ_L of $\dot{x} = A(t)x$ is the union of *Lyapunov spectral intervals*

$$\Sigma_L := \bigcup_{i=1}^{n} [\lambda_i^\ell, \lambda_i^u].$$

If each of the Lyapunov spectral intervals shrinks to a point, i.e., if $\lambda_i^\ell = \lambda_i^u \quad \forall i = 1, 2, ..., n$, then the system is called *Lyapunov-regular*. If a system is Lyapunov-regular, then we simply write λ_i for the Lyapunov exponents.

Definition

A change of variables $z = T^{-1}x$ with an invertible matrix function $T \in C^1(\mathbb{I}, \mathbb{R}^{n \times n})$ is called a *kinematic similarity transformation* if T and T^{-1} are bounded. If $\dot{T}$ is bounded as well, then it is called a *Lyapunov transformation*.

Theorem (Perron 1930)

For every linear ODE $\dot{x} = A(t)x$, there exists a Lyapunov transformation to upper triangular form, and this transformation can be chosen to be pointwise orthogonal.

Characterization of Lyapunov regularity

Theorem (Lyapunov 1907)

Let $B = [b_{i,j}] \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ *be bounded and upper-triangular. Then the system*

$$\dot{R} = B(t)R,$$

is Lyapunov-regular if and only if all the limits

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t b_{i,i}(s)ds, \quad i = 1, 2, ..., n,$$

exist and these limits coincide with the Lyapunov exponents $\lambda_i, \; i = 1, 2, ..., n$.

Stability of Lyapunov exponents

Definition

Consider a homogeneous linear system $\dot{x} = A(t)x$ with upper Lyapunov exponents λ_i^u and a perturbed system $\dot{x} = [A(t) + \Delta A(t)]x$ with upper Lyapunov exponents ν_i^u, both decreasingly ordered.

- The upper Lyapunov exponents, $\lambda_1^u \geq ... \geq \lambda_n^u$, are called *stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\sup_{t\geq 0} \|\Delta A(t)\| < \delta$ implies
$$|\lambda_i^u - \nu_i^u| < \epsilon, \quad i = 1, \ldots, n.$$
- A fundamental solution matrix X is called *integrally separated* if for $i = 1, 2, ..., n-1$, there exist $b > 0$ and $c > 0$ such that
$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq ce^{b(t-s)},$$
for all t, s with $t \geq s$.

Are Lyapunov exponents stable ?

Theorem (see e.g. Dieci/Van Vleck 2006)

i) *Integral separation is invariant under Lyapunov transformations (or kinematic similarity transformations).*

ii) *An integrally separated system has pairwise distinct upper (and pairwise distinct lower) Lyapunov exponents.*

iii) *Distinct upper Lyapunov exponents are stable if and only if there exists an integrally separated fundamental solution matrix.*

iv) *Integral separation is a generic property.*

Computation of Lyapunov exponents

Theorem (Dieci/Van Vleck 2002)

A system $\dot{x} = B(t)x$ with B bounded, continuous, and triangular, has an integrally separated fundamental solution matrix iff the diagonal elements of B are integrally separated.
If the diagonal of B is integrally separated, then $\Sigma_L = \Sigma_{CL}$, where

$$\Sigma_{CL} := \bigcup_{i=1}^{n} [\lambda_{i,i}^{\ell}, \lambda_{i,i}^{u}],$$

with

$$\lambda_{i,i}^{\ell} := \liminf_{t\to\infty} \frac{1}{t}\int_0^t b_{i,i}(s)ds, \;\; \lambda_{i,i}^{u} := \limsup_{t\to\infty} \frac{1}{t}\int_0^t b_{i,i}(s)ds, \;\; i = 1, 2, ..., n.$$

Bohl exponents, Piers Bohl 1865-1921

Definition

Let x be a nontrivial solution of $\dot{x} = A(t)x$. The *(upper) Bohl exponent* $\kappa_B^u(x)$ of this solution is the greatest lower bound of all those numbers ρ for which there exist numbers N_ρ such that

$$\|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\|$$

for any $t \geq s \geq 0$. If such numbers ρ do not exist, then one sets $\kappa_B^u(x) = +\infty$.
Similarly, the *lower Bohl exponent* $\kappa_B^\ell(x)$ is the least lower bound of all those numbers ρ' for which there exist numbers N'_ρ such that

$$\|x(t)\| \geq N'_\rho e^{\rho'(t-s)} \|x(s)\|, \quad 0 \leq s \leq t.$$

The interval $[\kappa_B^\ell(x), \kappa_B^u(x)]$ is called the *Bohl interval* of the solution.

Theorem (Daleckii/Krein 1974)

Bohl and Lyapunov exponents are related via

$$\kappa_B^\ell(x) \leq \lambda^\ell(x) \leq \lambda^u(x) \leq \kappa_B^u(x).$$

The Bohl exponents are given by

$$\kappa_B^u(x) = \limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}, \kappa_B^\ell(x) = \liminf_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}$$

If $A(t)$ is integrally bounded, i.e., if $\sup_{t\geq 0} \int_t^{t+1} \|A(s)\|\, ds < \infty$, then the Bohl exponents are finite.

Lyapunov and Bohl exponents

- Bohl exponents characterize the uniform growth rate of solutions, while Lyapunov exponents simply characterize the growth rate of solutions departing from $t = 0$.
- If the greatest bound of upper Lyapunov exponents for all solutions $\dot{x} = A(t)x$ is negative, then the system is asymptotically stable. If the same holds for the greatest upper bound of the upper Bohl exponents then the system is (uniformly) exponentially stable.
- Bohl exponents are stable without any extra assumption.

Definition

The fundamental matrix solution X of $\dot{X} = A(t)X$ is said to admit an *exponential dichotomy* if there exist a projector $P : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and constants $\alpha, \beta > 0$, as well as $K, L \geq 1$, such that

$$\begin{array}{rcll} \|X(t)PX^{-1}(s)\| & \leq & Ke^{-\alpha(t-s)}, & t \geq s, \\ \|X(t)(I-P)X^{-1}(s)\| & \leq & Le^{\beta(t-s)}, & t \leq s. \end{array}$$

The *Sacker-Sell (or exponential-dichotomy) spectrum* Σ_S for is given by those values $\lambda \in \mathbb{R}$ such that the *shifted system*

$$\dot{x}_\lambda = [A(t) - \lambda I]x_\lambda$$

does not have exponential dichotomy. The complement of Σ_S is called the *resolvent set*.

Properties of Sacker-Sell spectra

Theorem (Sacker/Sell 1978)

The property that a system possesses an exponential dichotomy as well as the exponential dichotomy spectrum are preserved under kinematic similarity transformations.

Σ_S *is the union of at most n disjoint closed intervals, and it is stable.*

Furthermore, the Sacker-Sell intervals contain the Lyapunov intervals, i.e.

$$\Sigma_L \subseteq \Sigma_S.$$

Characterization of Sacker-Sell spectra

Theorem (Dieci/Van Vleck 2006)

Consider $\dot{x} = B(t)x$ with B bounded, continuous, and upper triangular. The Sacker-Sell spectrum of this system and that of the corresponding diagonal system

$$\dot{x} = D(t)x, \quad \text{with } D(t) = \operatorname{diag}(b_{1,1}(t), \ldots, b_{n,n}(t)), \quad t \geq 0,$$

coincide.

Thus, one can retrieve Σ_S of $\dot{x} = A(t)x$ from the diagonal elements of the triangularized system.

Stability Theory for DAEs

- Lyapunov theory for regular constant coeff. DAEs Stykel 2002
- Index 1 systems Ascher/Petzold 1993,
- Systems of tractability index ≤ 2, Tischendorf 1994, Hanke/Macana/März 1998,
- Systems with properly stated leading term, Higueras/März/Tischendorf 2003, März 1998, März/Riazza 2002, Riazza 2002, Riazza/Tischendorf 2004.
- Lyapunov exponents and regularity, Cong/Nam 2003/2004.
- Exponential dichotomy in bound. val. problems, Lentini/März 1990.
- Exponential stability and Bohl exponents, Du/Linh 2006, 2007.
- General theory for linear DAEs Linh/M. 2008
- Perturbation theory Linh/M./Van Vleck 2009

We put $E(t)\dot{x} = A(t)x + f(t)$ and its derivatives up to order μ into a large DAE

$$M_k(t)\dot{z}_k = N_k(t)z_k + g_k(t), \quad k \in \mathbb{N}_0$$

for $z_k = (x, \dot{x}, \dots, x^{(k)})$.

$$M_2 = \begin{bmatrix} E & 0 & 0 \\ A - \dot{E} & E & 0 \\ \dot{A} - 2\ddot{E} & A - \dot{E} & E \end{bmatrix}, \; N_2 = \begin{bmatrix} A & 0 & 0 \\ \dot{A} & 0 & 0 \\ \ddot{A} & 0 & 0 \end{bmatrix}, \; z_2 = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}.$$

Theorem (Kunkel/M. 1996)

Under some constant rank assumptions, for a square regular linear DAE there exist integers μ, a, d such that:

1. $\operatorname{rank} M_\mu(t) = (\mu+1)n - a$ *on* $\mathbb{I}$, *and there exists a smooth matrix function* Z_2 *with* $Z_2^T M_\mu(t) = 0$.
2. *The first block column* $\hat{A}_2$ *of* $Z_2^* N_\mu(t)$ *has full rank* a *so that there exists a smooth matrix function* T_2 *such that* $\hat{A}_2 T_2 = 0$.
3. $\operatorname{rank} E(t) T_2 = d = n - a$ *and there exists a smooth matrix function* Z_1 *of size* (n, d) *with* $\operatorname{rank} \hat{E}_1 = d$, *where* $\hat{E}_1 = Z_1^T E$.

Reduced problem

- The quantity μ is called the strangeness-index. It describes the smoothness requirements for the inhomogeneity.
- It generalizes the usual differentiation index to general DAEs (and counts slightly differently).
- We obtain a numerically computable **strangeness-free formulation** of the equation with the same solution.

$$\begin{array}{rcll} \hat{E}_1(t)\dot{x} & = & \hat{A}_1(t)x + \hat{f}_1(t), & d \text{ differential equations} \\ 0 & = & \hat{A}_2(t)x + \hat{f}_2(t), & a \text{ algebraic equations} \end{array}$$

where $\hat{A}_1 = Z_1^T A$, $\hat{f}_1 = Z_1^T f$, and $\hat{f}_2 = Z_2^T g_\mu$.

- The reduced system is strangeness-free. This is a Remodelling! **For the theory we may assume that this has been done.**

Definition

A matrix function $X \in C^1(\mathbb{I}, \mathbb{R}^{n\times k})$, $\quad d \leq k \leq n$, is called *fundamental solution matrix of* $E(t)\dot{X} = A(t)X$ if each of its columns is a solution to $E(t)\dot{x} = A(t)x$ and $\operatorname{rank} X(t) = d$, for all $t \geq 0$.

A fundamental solution matrix is said to be *maximal* if $k = n$ and *minimal* if $k = d$, respectively.

A maximal fundamental matrix solution, denoted by $X(t, s)$, is called *normalized* if it satisfies the *projected initial condition* $E(0)(X(0, 0) - I) = 0$..

Every fundamental solution matrix has exactly d linearly independent columns and a minimal fundamental matrix solution can be easily made maximal by adding $n - d$ zero columns.

Definition

For a fundamental solution matrix X of a strangeness-free DAE system $E(t)\dot{x} = A(t)x$, and for $d \leq k \leq n$, we introduce

$$\lambda_i^u = \limsup_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\| \text{ and } \lambda_i^\ell = \liminf_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\|, \quad i = 1, 2, ..., k.$$

The $\lambda_i^u, i = 1, 2, ..., n$, are called *(upper) Lyapunov exponents* and the intervals $[\lambda_i^\ell, \lambda_i^u], i = 1, 2, ..., d$, are called *Lyapunov spectral intervals*. The DAE system is said to be *Lyapunov-regular* if $\lambda_i^l = \lambda_i^u, i = 1, 2, ..., d$.

Definition

Suppose that $W \in C(\mathbb{I}, \mathbb{R}^{n \times n})$ and $T \in C^1(\mathbb{I}, \mathbb{R}^{n \times n})$ are pointwise nonsingular matrix functions such that T and T^{-1} are bounded. Then the transformed DAE system

$$\tilde{E}(t)\dot{\tilde{x}} = \tilde{A}(t)\tilde{x},$$

with $\tilde{E} = WET$, $\tilde{A} = WAT - WE\dot{T}$ and $x = T\tilde{x}$ is called *globally kinematically equivalent* to $E(t)\dot{x} = A(t)x$. If, furthermore, also W and W^{-1} are bounded then we call this a *strong global kinematical equivalence transformation.*

Transformation to triangular form

Lemma

Consider a strangeness-free DAE $E(t)\dot{x} = A(t)x$ with continuous coefficients and a minimal fundamental solution matrix X. Then there exist pointwise orthogonal matrix functions *$U \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ and $V \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ such that in $E\dot{X} = AX$ the change of variables $X = VR$, with $R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ and $R_1 \in C^1(\mathbb{I}, \mathbb{R}^{d\times d})$ and the multiplication from the left with U^T leads to the system*

$$E_d \dot{R}_1 = A_d R_1,$$

where $E_d := U_1^T E V_1$ is pointwise nonsingular and $A_d := U_1^T A V_1 - U_1^T E \dot{V}_1$. Here U_1, V_1 are the matrix functions consisting of the first d columns of U, V, respectively.

Theorem

Consider a reduced strangeness-free DAE system. If the coefficient matrices are sufficiently smooth, then there exists an orthogonal matrix function $\hat{Q} \in C^1(\mathbb{I}, \mathbb{R}^{n \times n})$ such that with $\hat{x} = \hat{Q}^T x$, the submatrix E_1 is compressed, i.e., the transformed system has the form

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \geq 0,$$

Furthermore, this system is still strangeness-free and thus $\hat{E}_{11}$ and $\hat{A}_{22}$ are pointwise nonsingular.

The *associated underlying (implicit) ODE* is,

$$\hat{E}_{11} \dot{\hat{x}}_1 = \hat{A}_s \hat{x}_1,$$

where $\hat{A}_s := \hat{A}_{11} - \hat{A}_{12} \hat{A}_{22}^{-1} \hat{A}_{21}$.

Relation between Lyapunov exponents

Theorem

Let $\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21})$ be the upper Lyapunov exponent of the matrix function $\hat{A}_{22}^{-1}\hat{A}_{21}$. If the **boundedness condition**

$$\lambda^u(\hat{A}_{22}^{-1}\hat{A}_{21}) \leq 0$$

holds, then the upper Lyapunov exponents of the transformed DAE and its underlying ODE coincide if they are both ordered decreasingly.

Perturbed DAE systems

Consider perturbed DAEs

$$(E(t) + \Delta E(t))\dot{x} = (A(t) + \Delta A(t))x, \quad t \geq 0,$$

with *special perturbations* $(\Delta E, \Delta A)$, $\Delta E, \Delta A \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ that are sufficiently smooth and small enough such that by appropriate orthogonal transformation we obtain

$$\begin{bmatrix} \hat{E}_{11} + \Delta\hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} + \hat{A}_{11} & \hat{A}_{12} + \Delta\hat{A}_{12} \\ \hat{A}_{21} + \Delta\hat{A}_{21} & \hat{A}_{22} + \Delta\hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \geq 0.$$

If this is the case then we say that the perturbations are *admissible*.

Stability of Lyapunov exponents

Definition

The upper Lyapunov exponents $\lambda_1^u \geq ... \geq \lambda_d^u$ are said to be *stable* if for any $\epsilon > 0$, there exists $\delta > 0$ such that the conditions $\sup_t \left\|\Delta\tilde{E}(t)\right\| < \delta,\ \sup_t \left\|\Delta\tilde{A}(t)\right\| < \delta$ on admissable perturbations imply that the perturbed DAE system is strangeness-free and

$$|\lambda_i^u - \gamma_i^u| < \epsilon, \quad \forall i = 1, 2, ..., d,$$

where the γ_i^u are the ordered upper Lyapunov exponents of the perturbed system.
A DAE system and an admissably perturbed system are called *asymptotically equivalent* if they are strangeness-free and

$$\lim_{t\to\infty} \|\Delta E(t)\| = \lim_{t\to\infty} \|\Delta A(t)\| = 0.$$

Asymptotic equivalence

Theorem

Suppose that the DAE and an admissibly perturbed DAE are asymptotically equivalent. If the Lyapunov exponents are stable then $\lambda_i^u = \gamma_i^u$*, for all* $i = 1, 2, ..., d$*, where again the* γ_i^u *are the ordered upper Lyapunov exponents of the perturbed system.*

Definition

A minimal fundamental solution matrix X for a strangeness-free DAE is called *integrally separated* if for $i = 1, 2, ..., d - 1$ there exist $b > 0$ and $c > 0$ such that

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq ce^{b(t-s)},$$

for all t, s with $t \geq s \geq 0$.

Proposition

Consider a strangeness-free DAE.

1. *If the DAE is integrally separated then the same holds for any strongly globally kinematically equivalent system.*
2. *If the DAE is integrally separated, then it has pairwise distinct upper and pairwise distinct lower Lyapunov exponents.*
3. *If* $\hat{A}_{22}^{-1}\hat{A}_{21}$ *is bounded, then the DAE is integrally separated if and only if and the underlying ODE is integrally separated.*

Theorem

Consider a strangeness-free DAE and its transformed system, satisfying that

$$\hat{A}_{22}^{-1}\hat{A}_{21},\ \hat{A}_{12}\hat{A}_{22}^{-1},\ \hat{E}_{11},\ \hat{E}_{11}^{-1}(\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}),$$

are bounded. The DAE has d pairwise distinct upper and pairwise distinct lower Lyapunov exponents and they are stable iff it is integrally separated.
If $\hat{A}_{22}^{-1}\hat{A}_{21}$, $\hat{A}_{12}\hat{A}_{22}^{-1}$, $\hat{E}_{11}$, and $\hat{E}_{11}^{-1}$ are bounded, then the system has an integrally separated fundamental solution matrix iff its adjoint has.

Example For the DAE system

$$\begin{array}{rclrcl} \dot{x}_1 & = & x_1 + x_2, & 0 & = & x_1 - x_3, \\ \dot{x}_2 & = & x_2, & 0 & = & x_2 - e^{-t}x_4, \end{array}$$

the underlying ODE is not integrally separated, but the Lyapunov exponents are stable and the minimal fundamental solution

$$X(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \\ e^t & te^t \\ 0 & e^{2t} \end{bmatrix},$$

is integrally separated. However, the Lyapunov exponents of the DAE $(\{1,2\})$, are not stable.

Definition

Consider a strangeness-free DAE. For a scalar $\lambda \in \mathbb{R}$, the DAE system

$$E(t)\dot{x} = [A(t) - \lambda E(t)]x, \quad t \geq 0,$$

is called a *shifted DAE system*.
The shifted DAE transforms as

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \dot{\hat{x}} = \begin{bmatrix} \hat{A}_{11} - \lambda \hat{E}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \hat{x}, \quad t \geq 0,$$

and clearly, the shifted DAE system inherits the strangeness-free property from the original DAE.

Definition

A strangeness-free DAE system is said to have an *exponential dichotomy* if for a maximal fundamental solution $\hat{X}(t)$, there exists a projection matrix $P \in \mathbb{R}^{d\times d}$ and constants $\alpha, \beta > 0$, and $K, L \geq 1$ such that

$$\left\| \hat{X}(t) \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^{-}(s) \right\| \leq K e^{-\alpha(t-s)}, \quad t \geq s,$$
$$\left\| \hat{X}(t) \begin{bmatrix} I_d - P & 0 \\ 0 & 0 \end{bmatrix} \hat{X}^{-}(s) \right\| \leq L e^{\beta(t-s)}, \quad t \leq s.$$

Here $X^{-}(s)$ is a reflexive generalized inverse.

Theorem

A strangeness-free DAE system has an exponential dichotomy if and only if in the transformed system $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded and the underlying ODE has an exponential dichotomy.

Definition

- The *Sacker-Sell (or exponential dichotomy) spectrum* of a strangeness-free DAE system is defined by

$$\Sigma_S := \{\lambda \in \mathbb{R},\ \text{the shifted DAE does not have an exponential dichotomy}\}.$$

- The complement of Σ_S is called the *resolvent set*.
- The Sacker-Sell spectrum of a DAE system does not depend on the choice of the orthogonal change of basis that brings it to the transformed system.

Theorem

Consider a DAE and suppose that in the transformed DAE $\hat{A}_{22}^{-1}\hat{A}_{21}$ is bounded.

- *The Sacker-Sell spectrum is exactly the Sacker-Sell spectrum of the underlying ODE. It consists of at most d closed intervals.*
- *If the Sacker-Sell spectrum of the DAE system is given by d disjoint closed intervals, then there exists a minimal fundamental solution matrix $\hat{X}$ with integrally separated columns.*
- *In this case it is given exactly by the d (not necessarily disjoint) Bohl intervals associated with the columns of $\hat{X}$ and $\Sigma_L \subseteq \Sigma_S$.*

Theorem

Consider a strangeness-free DAE system

$$\hat{A}_{22}^{-1}\hat{A}_{21},\ \hat{A}_{12}\hat{A}_{22}^{-1},\ \hat{E}_{11},\ \hat{E}_{11}^{-1}(\hat{A}_{11}-\hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}),$$

are bounded and for $k \leq d$, $\Sigma_S = \bigcup_{i=1}^{k}[a_i, b_i]$.
Consider an admissible perturbation. Let $\varepsilon > 0$ *be sufficiently small such that* $b_i + \varepsilon < a_{i+1} - \varepsilon$ *for some* i, $0 \leq i \leq k$. *For* $i = 0$ *and* $i = k$, *set* $b_0 = -\infty$ *and* $a_{k+1} = \infty$, *respectively. Then, there exists* $\delta > 0$ *such that the inequality*

$$\max\{\sup_t \|\Delta E(t)\|, \sup_t \|\Delta A(t)\|\} \leq \delta,$$

implies that the interval $(b_i + \varepsilon, a_{i+1} - \varepsilon)$ *is contained in the resolvent of the perturbed DAE system.*

Corollary

Let the assumptions of the last Theorem hold and let $\varepsilon > 0$ *be sufficiently small such that* $b_{i-1} + \varepsilon < a_i - \varepsilon < a_i \le b_i < b_i + \varepsilon < a_{i+1} - \varepsilon$, *for* $0 \le i \le k$. *For* $i = 0$ *and* $i = k$, *set* $b_0 = -\infty$ *and* $a_{k+1} = \infty$, *respectively. Then, there exists* $\delta > 0$ *such that the inequality*

$$\max\{\sup_t \left\|\Delta\tilde{E}(t)\right\|, \sup_t \left\|\Delta\tilde{A}(t)\right\|\} \le \delta,$$

implies that the Sacker-Sell interval $[a_i, b_i]$ *either remains a Sacker-Sell interval under the perturbation or it is possibly split into several new intervals, but the smallest left end-point and the largest right end-point stay in the interval* $[a_i - \varepsilon, b_i + \varepsilon]$.

For the numerical computation we have first have to obtain the strangeness-free form of $E(t)$, $A(t)$.

It can be obtained pointwise for every t via the `FORTRAN` code `GELDA` Kunkel/M./Rath/Weickert 1997 or the corresponding `MATLAB` version Kunkel/M./Seidel 2005.

It comes in the form

$$\begin{bmatrix} E_1(t) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix} x$$

where A_2 is full row rank.

Smooth QR factorizations

Kunkel/M. 1991, Dieci/Eirola 1999 Suppose that the original DAE has sufficiently smooth coefficients.

- There exists a pointwise nonsingular, upper triangular matrix function $\tilde{A}_{22} \in C^1(\mathbb{I}, \mathbb{R}^{(n-d)\times(n-d)})$ and a pointwise orthogonal matrix function $\tilde{Q} \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ such that

$$A_2 = \begin{bmatrix} 0 & \tilde{A}_{22} \end{bmatrix} \hat{Q}.$$

- To make the factorization unique and to obtain smoothness, we require the diagonal elements of $\tilde{A}_{22}$ to be positive.
- Alternatively we can derive differential equations for $\tilde{Q}$ (or its Householder factors) and to solve the corresponding initial value problems.

The transformation $\tilde{x} = \tilde{Q}^T x$ leads to

$$\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \tilde{Q}, \quad \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tilde{Q} - \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \dot{\tilde{Q}}.$$

To evaluate $\dot{\tilde{Q}}$ at time t, we may use either a finite difference formula or the smooth QR method derived in Kunkel/M. 1991. The solution component $\tilde{x}_2$ associated with the algebraic equations is simply 0, thus we only have to deal with $\tilde{x}_1$.

Basic Idea for Computing Exponents

- Determine for every point t

  $\mathcal{E} = [e_{ij}] = P^T \tilde{E}_{11} Q,\ \mathcal{A} = [a_{ij}] = P^T \tilde{A}_{11} Q - P^T \tilde{E}_{11} \dot{Q} = P^T \tilde{A}_{11} Q - \mathcal{E} Q^T \dot{Q}$

  such that $\mathcal{E}$, $\mathcal{A}$ are upper triangular.

- Determine strictly lower triangular part of the skew symmetric $S(Q) = Q^T \dot{Q}$ by corresponding part of $\mathcal{E}^{-1} P^T \tilde{A}_{11} Q$ and the remaining part by skew-symmetry.

- Determine P and $\mathcal{E}$ via a smooth QR-factorization $\tilde{E}_{11} Q = P\mathcal{E}$.

- Keep orthogonality via orthogonal integrators Hairer/Lubich/Wanner 2002 or projected ODE integrators Dieci/Van Vleck 2003.

- Compute the spectral intervals from

$$\mathcal{E}_1(t)\dot{R}_1 = \mathcal{A}_1(t)R_1, \quad t \in \mathbb{I},$$

  where R_1 is the fundamental solution matrix of the triangularized underlying implicit ODE.

Continuous QR algorithm

- Compute a smooth QR factorization of A_2

$$\begin{bmatrix} \mathcal{E}_1 & U_1^T \tilde{E}_{12} \\ 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \mathcal{A}_1 & U_1^T \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} z,$$

  with $\mathcal{E}_1$, $\mathcal{A}_1$, $\tilde{A}_{22}$ upper triangular.

- Apply ODE methods of Dieci/Van Vleck to

$$\mathcal{E}_1(t)\dot{R}_1 = \mathcal{A}_1(t)R_1, \quad t \in \mathbb{I},$$

- Compute

$$\lambda_i(t_j) = \frac{1}{t_j} \ln[R_1(t_j)]_{i,i} = \frac{1}{t_j} \ln \prod_{\ell=1}^{j} [\Theta_\ell]_{i,i} = \frac{1}{t_j} \sum_{\ell=1}^{j} \ln[\Theta_\ell]_{i,i}, \quad i = 1, 2, \ldots, d.$$

- Solve optimization problems $\inf_{\tau \le t \le T} \lambda_i(t)$ and $\sup_{\tau \le t \le T} \lambda_i(t)$, $i = 1, 2, \ldots, d$ for a given $\tau \in (0, T)$.

Discrete QR algorithm

- Take a mesh $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$.
- Compute the fundamental solution $X^{[j]}$ on $[t_{j-1}, t_j]$ by solving
$$E\dot{X}^{[j]} = AX^{[j]},\ t_{j-1} \leq t \leq t_j,\ X^{[j]}(t_{j-1}) = \chi_{j-1},$$
using the DAE integrator `GELDA`.
- Determine QR factorizations
$$\tilde{Q}(t_j)^T X^{[j]}(t_j) = \begin{bmatrix} Z_j \\ 0 \end{bmatrix}, Z_j = Q_j \Theta_j, \quad j = 1, 2, \ldots, N$$
- Compute
$$\lambda_i(t_j) = \frac{1}{t_j} \ln[R_1(t_j)]_{i,i} = \frac{1}{t_j} \ln \prod_{\ell=1}^{j} [\Theta_\ell]_{i,i} = \frac{1}{t_j} \sum_{\ell=1}^{j} \ln[\Theta_\ell]_{i,i}, \quad i = 1, 2, \ldots, d,$$
- Solve the optimization problems $\inf_{\tau \leq t \leq T} \lambda_i(t)$ and $\sup_{\tau \leq t \leq T} \lambda_i(t)$, $i = 1, 2, \ldots, d$ for a given $\tau \in (0, T)$.

Lyapunov exp. via cont. QR-Euler method

Lyapunov regular 2×2 DAE $\lambda_1 = 5, \lambda_2 = 0$.

T	h	λ_1	λ_2	$CPU(s)$
500	0.1	4.9341	-0.0043	2.55
500	0.05	4.9337	-0.0038	5.01
500	0.01	4.9337	-0.0037	24.89
1000	0.1	4.9632	-0.0006	5.01
1000	0.05	4.9628	-0.0001	10.01
1000	0.01	4.9627	-0.0001	49.84
2000	0.1	4.9799	-0.0010	10.17
2000	0.05	4.9794	-0.0005	20.02
10000	0.1	4.9956	-0.0009	49.91
10000	0.05	4.9951	-0.0003	99.71

Lyapunov exp. via disc. QR-Euler method

Lyapunov regular 2×2 DAE $\lambda_1 = 5, \lambda_2 = 0$.

T	h	λ_1	λ_2	$CPU(s)$
500	0.1	5.0324	-0.0137	9.87
500	0.05	4.9818	-0.0087	19.59
500	0.01	4.9431	-0.0047	97.31
1000	0.1	5.0625	-0.0100	19.63
1000	0.05	5.0114	-0.0050	38.87
2000	0.1	5.0799	-0.0104	39.20
2000	0.05	5.0284	-0.0053	78.15
10000	0.1	5.0963	-0.0102	194.89
10000	0.05	5.0443	-0.0052	389.64

Lyapunov exp. via cont. QR-Euler method

Lyapunov non regular 2×2 DAE with Lyapunov intervals $[-1, 1]$, $[-6, -4]$.

T	t_0	h	$[\lambda_1^l, \lambda_1^u]$	$[\lambda_2^l, \lambda_2^u]$	$CPU(s)$
1000	100	0.1	$[-1.0018, 0.5865]$	$[-6.0006, -4.8928]$	5.3
5000	100	0.1	$[-1.0018, 1.0004]$	$[-6.0006, -4.3846]$	26.0
10000	100	0.1	$[-1.0018, 1.0004]$	$[-6.0006, -4.0235]$	51.5
10000	500	0.1	$[-0.0647, 1.0004]$	$[-6.0006, -4.0235]$	51.6
10000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0229]$	103.4
20000	100	0.1	$[-1.0018, 1.0004]$	$[-6.0006, -4.0007]$	103.5
20000	500	0.1	$[-0.4598, 1.0004]$	$[-6.0006, -4.0007]$	103.3
20000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0001]$	211.0
50000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0001]$	519.5
50000	500	0.05	$[-0.9844, 1.0000]$	$[-6.0001, -4.0001]$	518.2
100000	100	0.05	$[-1.0028, 1.0000]$	$[-6.0001, -4.0001]$	1044.9
100000	500	0.05	$[-0.9998, 1.0000]$	$[-6.0001, -4.0001]$	1050.4

Sacker-Sell intervals $[-\sqrt{2}, \sqrt{2}]$ and $[-5-\sqrt{2}, -5+\sqrt{2}]$.

T	H	h	$[\kappa_1^l, \kappa_1^u]$	$[\kappa_2^l, \kappa_2^u]$	$CPU(s)$
1000	100	0.1	$[-1.2042, 1.3811]$	$[-6.4049, -4.8927]$	6.2
5000	100	0.1	$[-1.2042, 1.4131]$	$[-6.4049, -3.5990]$	30.8
10000	100	0.1	$[-1.2042, 1.4131]$	$[-6.4049, -3.5867]$	61.9
10000	500	0.1	$[-0.7327, 1.4030]$	$[-6.2142, -3.5872]$	94.8
10000	100	0.05	$[-1.2049, 1.4127]$	$[-6.4046, -3.5860]$	147.2
20000	100	0.1	$[-1.3461, 1.4131]$	$[-6.4049, -3.5867]$	123.6
20000	500	0.1	$[-1.3416, 1.4030]$	$[-6.2142, -3.5872]$	201.3
20000	100	0.05	$[-1.3468, 1.4127]$	$[-6.4046, -3.5860]$	283.1
50000	100	0.1	$[-1.4132, 1.4131]$	$[-6.4049, -3.5867]$	310.4
50000	500	0.1	$[-1.4132, 1.4030]$	$[-6.2142, -3.5872]$	506.7
100000	100	0.1	$[-1.4132, 1.4131]$	$[-6.4049, -3.5867]$	646.2
100000	500	0.1	$[-1.4132, 1.4030]$	$[-6.3633, -3.5872]$	976.3
200000	500	0.1	$[-1.4132, 1.4030]$	$[-6.4147, -3.5872]$	1973.4

Conclusions

- The classical Theory of Lyapunov/Bohl/Sacker-Sell has been extended to linear DAEs with variable coefficients.
- Boundedness conditions and strangeness-free formulations are the key tools.
- In principle we can compute spectral intervals.
- Numerical methods for computing the Sacker-Sell spectra of DAEs extending work of Dieci/Van Vleck. They are **expensive**.
- Perturbation theory of Dieci/Van Vleck for ODEs has been extended and the methods have been modified to deal only with partial exponents (e.g. the largest Sacker-Sell interval) Linh/M./Van Vleck Dec. 2009
- SVD based methods for more accurate spectra.

Future work

- Extension to analysis of nonlinear systems
- Production code for classes special classes of DAEs
- Extension to operator DAEs/PDEs

Thank you very much
for your attention.