Preservers of Eigenvalue Inclusion Sets

Virginia Forstall, Aaron Herman, Chi-Kwong Li
Nung-Sing Sze, Vincent Yannello

May 24, 2010

Preserver problems on matrices

- Matrix preserver problems characterize maps on matrices leaving invariant a function, subset, or relation.
- Given a function f on a matrix set M with a binary operator $A \circ B$, maps $\Phi : M \mapsto M$ have been studied that satisfy $f(\Phi(A) \circ \Phi(B)) = f(A \circ B)$ for all $A, B \in M$.

Preserver problems on matrices

- Matrix preserver problems characterize maps on matrices leaving invariant a function, subset, or relation.
- Given a function f on a matrix set M with a binary operator $A \circ B$, maps $\Phi : M \mapsto M$ have been studied that satisfy $f(\Phi(A) \circ \Phi(B)) = f(A \circ B)$ for all $A, B \in M$.

Preservers of the spectrum of matrix prodcuts

- Many applications require knowledge of the eigenvalues of products or powers of matrices.
- For example, several applications in wavelet analysis require the joint spectral radius, which is the maximum eigenvalue of matrix products over a set of matrices [3].
- Researchers considered mappings that leave the spectrum of matrix products invariant.
- It has been shown that a map, $\Phi : M_n \mapsto M_n$, such that $Sp(\Phi(A)\Phi(B)) = Sp(AB)$ has the form

$$A \mapsto \pm S^{-1}AS \quad \text{or} \quad A \mapsto \pm S^{-1}A^tS$$

where $Sp(A)$ is the spectrum of A and S is an invertible matrix [1].

Preservers of the spectrum of matrix prodcuts

- Many applications require knowledge of the eigenvalues of products or powers of matrices.
- For example, several applications in wavelet analysis require the joint spectral radius, which is the maximum eigenvalue of matrix products over a set of matrices [3].
- Researchers considered mappings that leave the spectrum of matrix products invariant.
- It has been shown that a map, $\Phi : M_n \mapsto M_n$, such that $Sp(\Phi(A)\Phi(B)) = Sp(AB)$ has the form

$$A \mapsto \pm S^{-1}AS \quad \text{or} \quad A \mapsto \pm S^{-1}A^tS$$

where $Sp(A)$ is the spectrum of A and S is an invertible matrix [1].

Preservers of the spectrum of matrix prodcuts

- Many applications require knowledge of the eigenvalues of products or powers of matrices.
- For example, several applications in wavelet analysis require the joint spectral radius, which is the maximum eigenvalue of matrix products over a set of matrices [3].
- Researchers considered mappings that leave the spectrum of matrix products invariant.
- It has been shown that a map, $\Phi : M_n \mapsto M_n$, such that $Sp(\Phi(A)\Phi(B)) = Sp(AB)$ has the form

$$A \mapsto \pm S^{-1}AS \quad \text{or} \quad A \mapsto \pm S^{-1}A^tS$$

where $Sp(A)$ is the spectrum of A and S is an invertible matrix [1].

Preservers of the spectrum of matrix prodcuts

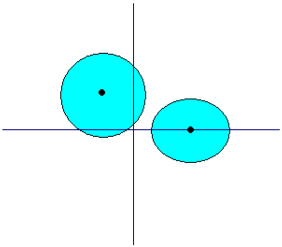
- Many applications require knowledge of the eigenvalues of products or powers of matrices.
- For example, several applications in wavelet analysis require the joint spectral radius, which is the maximum eigenvalue of matrix products over a set of matrices [3].
- Researchers considered mappings that leave the spectrum of matrix products invariant.
- It has been shown that a map, $\Phi : M_n \mapsto M_n$, such that $Sp(\Phi(A)\Phi(B)) = Sp(AB)$ has the form

$$A \mapsto \pm S^{-1}AS \quad \text{or} \quad A \mapsto \pm S^{-1}A^tS$$

  where $Sp(A)$ is the spectrum of A and S is an invertible matrix [1].

Eigenvalue Inclusion Sets

- In applications, it may not be easy to determine the eigenvalues precisely due to numerical error, insufficient information, size of matrices, etc.
- Even without the exact eigenvalues, we have a lot of information about the matrix by knowing its eigenvalue inclusion set.



Preservers of eigenvalue inclusion sets of matrix products

- We consider maps that preserve eigenvalue containment regions for products of matrices.
- Let $\mathcal{S}(X)$ be an eigenvalue inclusion set of the matrix $X \in M_n$. Consider a preserver, $\Phi : M_n \mapsto M_n$, of the eigenvalue inclusion set of matrix products such that $\mathcal{S}(\Phi(A)\Phi(B)) = \mathcal{S}(AB)$ for all $A, B \in M_n$.
- We characterize such maps when $\mathcal{S}(A)$ is the Gershgorin, Brauer, and Ostrowski region of $A \in M_n$.

Preservers of eigenvalue inclusion sets of matrix products

- We consider maps that preserve eigenvalue containment regions for products of matrices.
- Let $\mathcal{S}(X)$ be an eigenvalue inclusion set of the matrix $X \in M_n$. Consider a preserver, $\Phi : M_n \mapsto M_n$, of the eigenvalue inclusion set of matrix products such that $\mathcal{S}(\Phi(A)\Phi(B)) = \mathcal{S}(AB)$ for all $A, B \in M_n$.
- We characterize such maps when $\mathcal{S}(A)$ is the Gershgorin, Brauer, and Ostrowski region of $A \in M_n$.

Preservers of eigenvalue inclusion sets of matrix products

- We consider maps that preserve eigenvalue containment regions for products of matrices.
- Let $\mathcal{S}(X)$ be an eigenvalue inclusion set of the matrix $X \in M_n$. Consider a preserver, $\Phi : M_n \mapsto M_n$, of the eigenvalue inclusion set of matrix products such that $\mathcal{S}(\Phi(A)\Phi(B)) = \mathcal{S}(AB)$ for all $A, B \in M_n$.
- We characterize such maps when $\mathcal{S}(A)$ is the Gershgorin, Brauer, and Ostrowski region of $A \in M_n$.

Gershgorin region

- The Gershgorin region is the most common eigenvalue inclusion set.
- A Gershgorin disc is defined by

$$G_i(A) = \{\mu \in C : |\mu - a_{ii}| \leq R_i\}$$

where R_i is the deleted row sum.
- The Gershgorin region is the union of these disks and includes all the eigenvalues of A.

Gershgorin region

- The Gershgorin region is the most common eigenvalue inclusion set.
- A Gershgorin disc is defined by

$$G_i(A) = \{\mu \in C : |\mu - a_{ii}| \leq R_i\}$$

where R_i is the deleted row sum.
- The Gershgorin region is the union of these disks and includes all the eigenvalues of A.

Gershgorin region

- The Gershgorin region is the most common eigenvalue inclusion set.
- A Gershgorin disc is defined by

$$G_i(A) = \{\mu \in C : |\mu - a_{ii}| \leq R_i\}$$

  where R_i is the deleted row sum.
- The Gershgorin region is the union of these disks and includes all the eigenvalues of A.

Ostrowski's region

Definition

For $\epsilon \in [0,1]$, define the Ostrowski region of $A \in M_n$ such that $O_\epsilon(A) = \cup_{j=1}^n O_{\epsilon,j}$, where

$$O_{\epsilon,j} = \{\mu \in C | \mu - a_{jj}| \leq R_j^\epsilon C_j^{1-\epsilon}\},$$

and $R_j = \sum_{k \neq j} |a_{jk}|$ and $C_j = \sum_{k \neq j} |a_{kj}|$.

- The Ostrowski region reduces to the Gershgorin region when $\epsilon = 1$ and the Gershgorin region of A^t when $\epsilon = 0$.

Ostrowski's region

Definition

For $\epsilon \in [0, 1]$, define the Ostrowski region of $A \in M_n$ such that $O_\epsilon(A) = \cup_{j=1}^{n} O_{\epsilon,j}$, where

$$O_{\epsilon,j} = \{\mu \in C | \mu - a_{jj}| \leq R_j^{\epsilon} C_j^{1-\epsilon}\},$$

and $R_j = \sum_{k \neq j} |a_{jk}|$ and $C_j = \sum_{k \neq j} |a_{kj}|$.

- The Ostrowski region reduces to the Gershgorin region when $\epsilon = 1$ and the Gershgorin region of A^t when $\epsilon = 0$.

Example:

Consider a matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -i & 1 \\ 0 & 0 & i \end{pmatrix}$

- The Ostrowski region is smaller than the Gershgorin region when $\epsilon \in (0, 1)$.

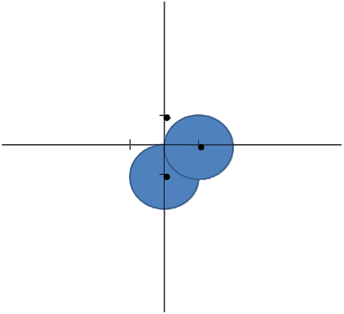


Figure: $G(A)$

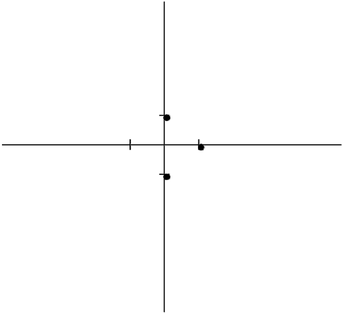


Figure: $O_\epsilon(A)$ for $\epsilon \in (0, 1)$

Preservers of Ostrowski regions

For the Ostrowski region, we have the following.

Theorem

A mapping $\Phi : M_n \mapsto M_n$ satisfies

$$O_\epsilon(\Phi(A)\Phi(B)) = O_\epsilon(AB), \forall A, B \in M_n$$

if and only if Φ has the form

$$A \mapsto \pm(DP)A(DP)^{-1}$$

where P is a permutation matrix and D is diagonal and invertible. D is unitary except when $(n, \epsilon) = (2, 1/2)$.

Outline of Proof

- It is easy to check that a map of this form preserves the Ostrowski set.
- Proving the converse is more involved.
- The key step is to show that there exists P such that

$$P\Phi(E_{ij})P^t = \nu_{ij}E_{ij}$$

with $\nu_{ij} \neq 0$ where E_{ij} is the standard basis for M_n.

Outline of Proof

- It is easy to check that a map of this form preserves the Ostrowski set.
- Proving the converse is more involved.
- The key step is to show that there exists P such that

$$P\Phi(E_{ij})P^t = \nu_{ij}E_{ij}$$

with $\nu_{ij} \neq 0$ where E_{ij} is the standard basis for M_n.

Outline of Proof

- It is easy to check that a map of this form preserves the Ostrowski set.
- Proving the converse is more involved.
- The key step is to show that there exists P such that

$$P\Phi(E_{ij})P^t = \nu_{ij}E_{ij}$$

with $\nu_{ij} \neq 0$ where E_{ij} is the standard basis for M_n.

Useful ideas

- Considering $O_\epsilon(A^2) = O_\epsilon(\Phi(A)^2)$ for certain $A \in M_n$ yields information about the structure of $\Phi(A)^2$.
- Taking A with n distinct eigenvalues and an eigenvalue inclusion region of n degenerate discs, we can translate this information about $\Phi(A)^2$ to the structure $\Phi(A)$ using a known theorem.

Useful ideas

- Considering $O_\epsilon(A^2) = O_\epsilon(\Phi(A)^2)$ for certain $A \in M_n$ yields information about the structure of $\Phi(A)^2$.
- Taking A with n distinct eigenvalues and an eigenvalue inclusion region of n degenerate discs, we can translate this information about $\Phi(A)^2$ to the structure $\Phi(A)$ using a known theorem.

Useful results

Theorem

For $A \in M_n$ with n distinct eigenvalues, $B \in M_n$ commutes with A if and only if there is a (complex) polynomial of degree at most $n-1$ such that $B = p(A)$.

- Specifically, a diagonal matrix $(\Phi(A))^2 \in M_n$ with n distinct eigenvalues commutes with $\Phi(A)$, so $\Phi(A)$ can be written as a polynomial of $\Phi(A)^2$.
- So if we take A such that $O(A^2)$ has n degenerate disks, we conclude that $\Phi(A)$ a particular form (either diagonal, upper triangular etc.).

Useful results

Theorem

For $A \in M_n$ with n distinct eigenvalues, $B \in M_n$ commutes with A if and only if there is a (complex) polynomial of degree at most $n-1$ such that $B = p(A)$.

- Specifically, a diagonal matrix $(\Phi(A))^2 \in M_n$ with n distinct eigenvalues commutes with $\Phi(A)$, so $\Phi(A)$ can be written as a polynomial of $\Phi(A)^2$.
- So if we take A such that $O(A^2)$ has n degenerate disks, we conclude that $\Phi(A)$ a particular form (either diagonal, upper triangular etc.).

Useful results

Theorem

For $A \in M_n$ with n distinct eigenvalues, $B \in M_n$ commutes with A if and only if there is a (complex) polynomial of degree at most $n-1$ such that $B = p(A)$.

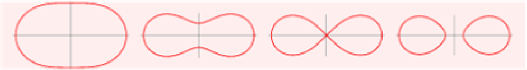
- Specifically, a diagonal matrix $(\Phi(A))^2 \in M_n$ with n distinct eigenvalues commutes with $\Phi(A)$, so $\Phi(A)$ can be written as a polynomial of $\Phi(A)^2$.
- So if we take A such that $O(A^2)$ has n degenerate disks, we conclude that $\Phi(A)$ a particular form (either diagonal, upper triangular etc.).

Brauer's region

Definition

The Brauer set of $A \in M_n$ is $C(1) = \cup_{1 \le i < j \le n} C_{ij}(A)$, where

$$C_{ij}(A) = \{\mu \in C : |(\mu - a_{ii})(\mu - a_{jj})| \le R_i R_j\}.$$



Preservers of Brauer's region

- We have a similar theorem for this containment region.

Theorem

A mapping $\Phi : M_n \mapsto M_n$ satisfies

$$C(\Phi(A)\Phi(B)) = C(AB), \forall A, B \in M_n$$

if and only if Φ has the form

$$A \mapsto \pm(DP)A(DP)^{-1}$$

where P is a permutation matrix and D is diagonal and invertible. The matrix D is unitary for $n \geq 3$.

- The proof of this theorem makes use of the theorem about commuting matrices to show that $\Phi : M_n \mapsto M_n$ maps $E_{ij} \mapsto \nu_{ij} E_{rs}$ for nonzero ν_{ij}.

Preservers of Brauer's region

- We have a similar theorem for this containment region.

Theorem

A mapping $\Phi : M_n \mapsto M_n$ satisfies

$$C(\Phi(A)\Phi(B)) = C(AB), \forall A, B \in M_n$$

if and only if Φ has the form

$$A \mapsto \pm(DP)A(DP)^{-1}$$

where P is a permutation matrix and D is diagonal and invertible. The matrix D is unitary for $n \geq 3$.

- The proof of this theorem makes use of the theorem about commuting matrices to show that $\Phi : M_n \mapsto M_n$ maps $E_{ij} \mapsto \nu_{ij} E_{rs}$ for nonzero ν_{ij}.

Preservers of Brauer's region

- We have a similar theorem for this containment region.

Theorem

A mapping $\Phi : M_n \mapsto M_n$ satisfies

$$C(\Phi(A)\Phi(B)) = C(AB), \forall A, B \in M_n$$

if and only if Φ has the form

$$A \mapsto \pm(DP)A(DP)^{-1}$$

where P is a permutation matrix and D is diagonal and invertible. The matrix D is unitary for $n \geq 3$.

- The proof of this theorem makes use of the theorem about commuting matrices to show that $\Phi : M_n \mapsto M_n$ maps $E_{ij} \mapsto \nu_{ij} E_{rs}$ for nonzero ν_{ij}.

Conclusion

- We have characterized maps that preserve eigenvalue inclusion sets such that $A \mapsto \pm(DP)A(DP)^{-1}$.
- Next we would like to characterize maps that satisfy $\mathcal{S}(\Phi(A) \circ \Phi(B)) = \mathcal{S}(A \circ B)$ for other eigenvalue inclusion sets $\mathcal{S}$ and other types of binary operation $\circ$ on matrices such as the Jordan product: $A \circ B = AB + BA$, or the Lie product: $A \circ B = AB - BA$.

Conclusion

- We have characterized maps that preserve eigenvalue inclusion sets such that $A \mapsto \pm(DP)A(DP)^{-1}$.
- Next we would like to characterize maps that satisfy $\mathcal{S}(\Phi(A) \circ \Phi(B)) = \mathcal{S}(A \circ B)$ for other eigenvalue inclusion sets $\mathcal{S}$ and other types of binary operation $\circ$ on matrices such as the Jordan product: $A \circ B = AB + BA$, or the Lie product: $A \circ B = AB - BA$.

References

- C.K. Li, P. Šemrl and N.S. Sze, Maps preserving the nilpotency of products of operators, Linear Algebra Appl. 424 (2007), 222–239.
- R.A. Horn and C.R.Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
- J. Theys, Joint Spectral Radius: Theory and Approximations, Ph.D Thesis, Universite catholique de Louvain, 2005.

Questions

Questions?

Thank You!

Questions

Questions?

Thank You!