A numerical range for rectangular matrices and matrix polynomials

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DEFINITIONS

The **(standard) numerical range** of a square matrix $A \in \mathbb{C}^{n \times n}$ is defined by

$$F(A) = \left\{ x^* A x \in \mathbb{C} : x \in \mathbb{C}^n, \ \|x\|_2 = \sqrt{x^* x} = 1 \right\}.$$

$F(A)$ is a **compact** and **convex** subset of $\mathbb{C}$ that contains the eigenvalues of A and has interesting geometric properties.

Since late 1920's (Toeplitz-Hausdorff Theorem), hundreds of papers have been published on the topic, all of them for square matrices.

Bonsall and Duncan (1973) observed that

$$\begin{aligned} F(A) &= \{\mu \in \mathbb{C} : \|A - \lambda I_n\|_2 \geq |\mu - \lambda|, \, \forall\, \lambda \in \mathbb{C}\} \\ &= \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \|A - \lambda I_n\|_2\right) \quad \text{(closed disks centered at } \lambda \text{)}. \end{aligned}$$

For any $A, B \in \mathbb{C}^{n \times m}$ and any matrix norm $\| \cdot \|$, we define the (**compact** and **convex**) **numerical range of** A **with respect to** B

$$\begin{aligned} F_{\|\cdot\|}(A; B) &= \{\mu \in \mathbb{C} : \|A - \lambda B\| \geq |\mu - \lambda|, \, \forall\, \lambda \in \mathbb{C}\} \\ &= \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \|A - \lambda B\|\right). \end{aligned}$$

For elements u, v of a normed linear space, u is **Birkhoff-James orthogonal** to v, $u \perp_{BJ} v$, if $\|u + \lambda v\| \geq \|u\|$, $\forall\, \lambda \in \mathbb{C}$.

We see that, in general,

$$F_{\|\cdot\|}(A; B) \supseteq \{\mu \in \mathbb{C} : B \perp_{BJ} (A - \mu B)\}\,,$$

and if $||B\| = 1$, then

$$F_{\|\cdot\|}(A; B) = \{\mu \in \mathbb{C} : B \perp_{BJ} (A - \mu B)\}\,.$$

WHY USING B ?

For any $A \in \mathbb{C}^{n\times n}$, $F(A) = \{\mu \in \mathbb{C} : \|A - \lambda I_n\|_2 \geq |\mu - \lambda|, \forall\, \lambda \in \mathbb{C}\}$.

In the rectangular case, one may question the use of B instead of $I_{n,m}$.

Without loss of generality, assume that $n > m$, $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ with

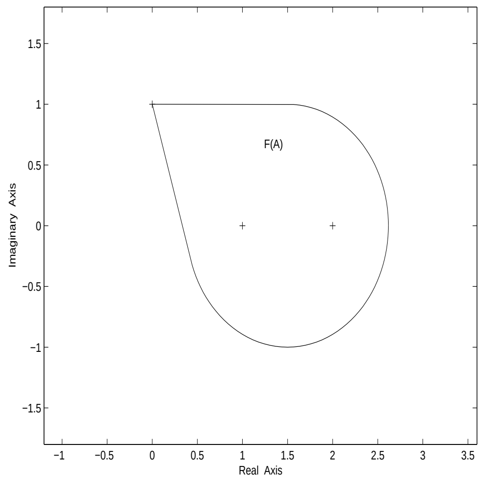
$A_1 \in \mathbb{C}^{m\times m}$ and $A_2 \in \mathbb{C}^{(n-m)\times m}$, and $I_{n,m} = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$.

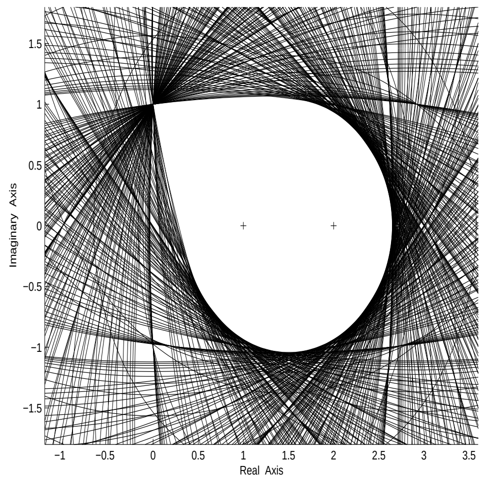
Theorem 1 $F_{\|\cdot\|_2}(A; I_{n,m}) = F(A_1)$.

BASIC PROPERTIES

We can estimate $F_{\|\cdot\|}(A;B) = \bigcap_{\lambda\in\mathbb{C}} \mathcal{D}\left(\lambda, \|A-\lambda B\|\right)$ by drawing circles $\partial\mathcal{D}\left(\lambda, \|A-\lambda B\|\right)$. To confirm the effectiveness of this procedure, let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \mathrm{i} \end{bmatrix}$$ and $B = I_3$, and recall that $F(A) = F_{\|\cdot\|_2}(A;I_3)$.





Proposition 2 $F_{\|\cdot\|}(A;B) \neq \emptyset \quad \Leftrightarrow \quad \|B\| \geq 1.$

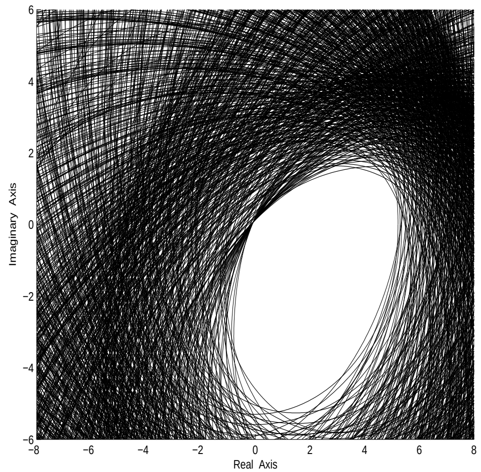
Proposition 3 If $a, b \in \mathbb{C}$

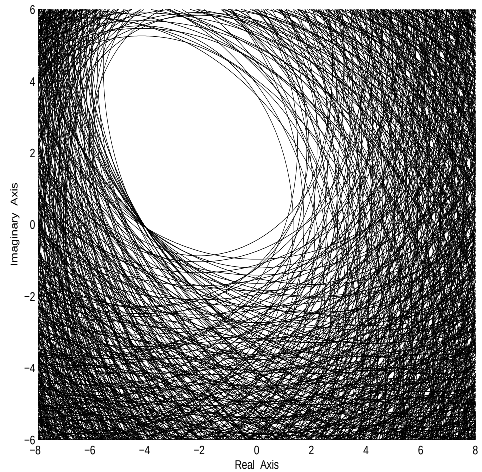
$$\Rightarrow \quad F_{\|\cdot\|}(bB;B) = \{b\} \text{ and } F_{\|\cdot\|}(aA + bB;B) = aF_{\|\cdot\|}(A;B) + b.$$

Proposition 4 If $\|\cdot\|$ is unitarily invariant, $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$ are unitary, and $\hat{A}, \hat{B}$ are associated submatrices of A, B

$$\Rightarrow \quad F_{\|\cdot\|}(UAV;UBV) = F_{\|\cdot\|}(A;B) \text{ and } F_{\|\cdot\|}(\hat{A};\hat{B}) \subseteq F_{\|\cdot\|}(A;B).$$

For $A = \begin{bmatrix} 5+\mathrm{i} & 0.2 & 0 & -0.1 \\ 0 & 1-\mathrm{i}5 & -\mathrm{i}0.1 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1.1 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,
$F_{\|\cdot\|_1}(A;B)$ and $F_{\|\cdot\|_1}(\mathrm{i}A-4B;B)$ in the left and right parts of the figure confirm the second part of Proposition 3.





Proposition 5 If $\|B\| > 1$ and $\mu_0 \in \vartheta F_{\|\cdot\|}(A;B)$,

$\Rightarrow \quad \exists\, \lambda_0 \in \mathbb{C}$ such that $\|A - \lambda_0 B\| = |\mu_0 - \lambda_0|$.

Corollary 6 If $\|B\| > 1 \quad \Rightarrow \quad \partial F_{\|\cdot\|}(A;B)$ has no flat portions.

Proposition 7 (Resolvent Estimate)
If $n = m$, B is invertible and $\|B^{-1}\| \leq 1$

$$\Rightarrow \quad d(\xi, F_{\|\cdot\|}(A;B)) \leq \frac{1}{\|(A - \xi B)^{-1}\|}, \quad \forall\, \xi \notin F_{\|\cdot\|}(A;B).$$

Proposition 8 If $\|\cdot\|$ is induced by the inner product $\langle\cdot,\cdot\rangle$

$$\Rightarrow \quad F_{\|\cdot\|}(A;B) = \mathcal{D}\left(\frac{\langle A,B\rangle}{\|B\|^2}, \left\|A - \frac{\langle A,B\rangle}{\|B\|^2}B\right\| \frac{\sqrt{\|B\|^2-1}}{\|B\|}\right).$$

Note that if $\|\cdot\|$ is induced by the inner product $\langle\cdot,\cdot\rangle$ and $\|B\|=1$

$\Rightarrow \quad F_{\|\cdot\|}(A;B) = \{\langle A,B\rangle\}$, i.e., it is a singleton, although A is not necessarily a scalar multiple of B.

EIGENVALUES

Let $A, B \in \mathbb{C}^{n\times m}$ with $n \geq m$, and $\| \cdot \|$ be induced by a vector norm.

A $\mu_0 \in \mathbb{C}$ is said to be an **eigenvalue of A with respect to B** if $(A - \mu_0 B)\, x_0 = 0$ for some $0 \neq x_0 \in \mathbb{C}^m$ (the **eigenvector**).

Proposition 9 If μ_0 is an eigenvalue of A with respect to B, with a unit eigenvector $x_0 \in \mathbb{C}^m$ such that $\|Bx_0\| \geq 1 \quad \Rightarrow \quad \mu_0 \in F_{\|\cdot\|}(A;B)$.

MATRIX POLYNOMIALS

Consider an $n \times m$ **matrix polynomial (m.p.)**

$$P(z) = A_l z^l + A_{l-1} z^{l-1} + \cdots + A_1 z + A_0,$$

where $z \in \mathbb{C}$ and $A_j \in \mathbb{C}^{n\times m}$ $(j = 0, 1, \ldots, l)$ with $A_l \neq 0$.

If $n \geq m$, then a $\mu_0 \in \mathbb{C}$ is an **eigenvalue** of $P(z)$ if $P(\mu_0)x_0 = 0$ for some $0 \neq x_0 \in \mathbb{C}^m$ (the **eigenvector**).

For $n = m$, the **(standard) numerical range** of m.p. $P(z)$ is

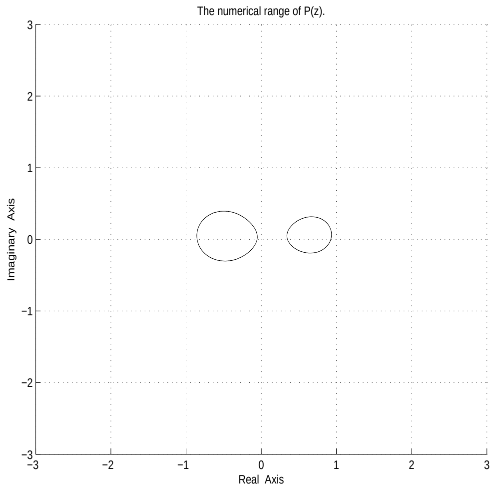
$$\begin{aligned} W(P(z)) &= \{\mu \in \mathbb{C} : x^* P(\mu) x = 0,\ x \in \mathbb{C}^n,\ x \neq 0\} \\ &= \{\mu \in \mathbb{C} : 0 \in F(P(\mu))\}\,. \end{aligned}$$

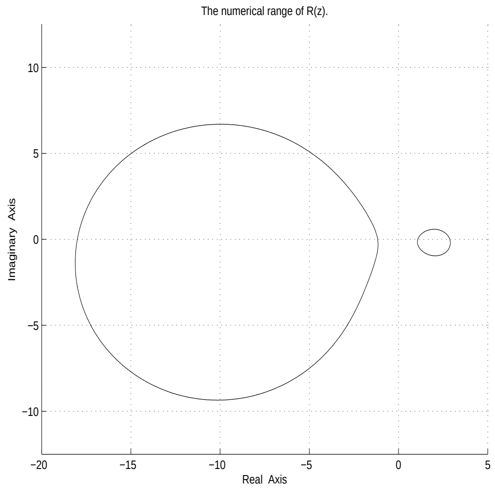
We define the **numerical range of** $P(z)$ **with respect to** B

$$\begin{aligned} W_{\|\cdot\|}(P(z);B) &= \{\mu \in \mathbb{C} : 0 \in F_{\|\cdot\|}(P(\mu);B)\} \\ &= \{\mu \in \mathbb{C} : \|P(\mu) - \lambda B\| \geq |\lambda|, \forall\, \lambda \in \mathbb{C}\}\,. \end{aligned}$$

The closeness of $W_{\|\cdot\|}(P(z);B)$ follows from the continuity of norms.

If $P(z) = Bz - A \;\Rightarrow\; W_{\|\cdot\|}(Bz - A;B) = F_{\|\cdot\|}(A;B)$.





The ranges $W_{\|\cdot\|_F}(P(z);B)$ and $W_{\|\cdot\|_F}(R(z);B)$ for $B = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}$,

$$P(z) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1.6 & 0 \end{bmatrix} z^2 + \begin{bmatrix} \mathrm{i} & 1 & -1 \\ 0 & 2 & \mathrm{i} \end{bmatrix} z + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \end{bmatrix} \text{ and}$$

$$R(z) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \end{bmatrix} z^2 + \begin{bmatrix} \mathrm{i} & 1 & -1 \\ 0 & 2 & \mathrm{i} \end{bmatrix} z + \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1.6 & 0 \end{bmatrix}.$$

BASIC PROPERTIES

Proposition 10 $W_{\|\cdot\|}(P(z);B) \neq \emptyset \quad \Leftrightarrow \quad \|B\| \geq 1.$

Proposition 11 If $R(z) = A_0 z^l + \cdots + A_{l-1} z + A_l = z^l P(z^{-1})$

$$\Rightarrow \quad W_{\|\cdot\|}(R(z);B) \backslash \{0\} = \left\{ z^{-1} : z \in W_{\|\cdot\|}(P(z);B) \backslash \{0\} \right\}.$$

Proposition 12 If norm $\|\cdot\|$ is induced by a vector norm, $n \geq m$, and μ_0 is an eigenvalue of $P(z)$ with an associated unit eigenvector $x_0 \in \mathbb{C}^n$ such that $\|Bx_0\| \geq 1 \quad \Rightarrow \quad \mu_0 \in W_{\|\cdot\|}(P(z);B)$.

Theorem 13 (i) If $W_{\|\cdot\|}(P(z);B)$ is unbounded $\Rightarrow$ $0 \in F_{\|\cdot\|}(A_l;B)$.

(ii) If $0 \in F_{\|\cdot\|}(A_l;B)$, 0 is not an isolated point of $W_{\|\cdot\|}(R(z);B)$

$\Rightarrow$ $W_{\|\cdot\|}(P(z);B)$ is unbounded.

Theorem 14 (i) If $\mu_0 \in \partial W_{\|\cdot\|}(P(z);B)$ $\Rightarrow$ $0 \in \partial F_{\|\cdot\|}(P(\mu_0);B)$.

(ii) If $0 \in \partial F_{\|\cdot\|}(P(\mu_0);B)$, $P(\mu_0) \neq 0$, $0 \notin F_{\|\cdot\|}(P'(\mu_0);B)$, $\|B\| > 1$

$\Rightarrow$ $\mu_0 \in \partial W_{\|\cdot\|}(P(z);B)$.

Proposition 15 If $\|B\| > 1$, and $\mu \in \mathbb{C}$ such that $P(\mu) = 0$ and $0 \notin F_{\|\cdot\|}(P'(\mu); B)$ $\Rightarrow$ μ is an isolated point of $W_{\|\cdot\|}(P(z); B)$.

Proposition 16 If norm $\|\cdot\|$ is induced by the inner product $\langle \cdot, \cdot \rangle$

$$\Rightarrow \quad W_{\|\cdot\|}(P(z); B) = \left\{ \mu \in \mathbb{C} : |\langle P(\mu), B \rangle| \leq \|P(\mu)\| \sqrt{\|B\|^2 - 1} \right\}.$$

Corollary 17 If norm $\|\cdot\|$ is induced by an inner product

$\Rightarrow$ the boundary of $W_{\|\cdot\|}(P(z); B)$ lies on an algebraic curve.

REFERENCES

F.F. Bonsall and J. Duncan, *Numerical Ranges II*, Cambridge University Press, 1973.

Ch. Chorianopoulos, S. Karanasios and P. Psarrakos, A definition of numerical range of rectangular matrices, *Linear and Multilinear Algebra*, **57** (2009).

Ch. Chorianopoulos and P. Psarrakos, A numerical range of rectangular matrices and matrix polynomials, *in preparation* (2010).

I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, Academic Press, 1982.

R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

R.C. James, Orthogonality and linear functionals in normed linear spaces, *Trans. Amer. Math. Soc.*, **61** (1947).

P. Lancaster and P. Psarrakos, Normal and seminormal eigenvalues of analytic matrix functions, *Integral Equations Operator Theory*, **41** (2001).

C.-K. Li and L. Rodman, Numerical range of matrix polynomials, *SIAM J. Matrix Anal. Appl.*, **15** (1994).

J. Maroulas and P. Psarrakos, The boundary of numerical range of matrix polynomials, *Linear Algebra Appl.*, **267** (1997).