MONOTONE SOLUTIONS
OF SOME LINEAR EQUATIONS
(IN BANACH SPACES)

Ivo Marek
Czech University of Technology in Prague

Sympozium
ALA 2010 in honor of Hans Schneider
Novi Sad, May 24-28, 2010

OUTLINE OF THE TALK

- A motivation example

- An application

- Analytical tools

- Basic result

- Some extensions

- Concluding remarks

1

1.1 Example *Let* $N \geq 1$ *and*

$$x_{k+N} = a_1 x_k + a_2 x_{k+1} + ... + a_N x_{k+N-1}, \quad k = 0, 1, ..., \tag{1.1}$$

where $a_1, ..., a_N$ *are arbitrary real numbers.*

Question 1: When does the above difference equation possess a strictly increasing solution?

1.2 Answer *Let* $N \geq 1$ *and*

$$x_{k+N} = a_1 x_k + a_2 x_{k+1} + ... + a_N x_{k+N-1},\ k = 0, 1, ..., \tag{1.2}$$

where $a_1, ..., a_N$ *are arbitrary real numbers. A strictly monotone solution of the given difference equation (5.4) exists if and only if the characteristic polynomial* p *of (5.4) possesses either a positive root* $\mu \neq 1$ *or* $\mu = 1$ *is a root of* p *whose multiplicity is at least two.*

Question 2: Who can be interested in strictly monotone solutions to difference equations shown on the previous page?

2

Bobok J. *On entropy of patterns given by interval maps.* Fundamenta Mathematicae **162**(1999), 1-36.

Bobok J., Kuchta M. *X-minimal patterns and generalization of Sharkovskii's Theorem.* Fundamenta Mathematicae **156**(1998), 33-66.

Concepts utilized in the field of Combinatorial Discrete Dynamical Systems:

Let $P = \{1, ..., n\} \subset \mathcal{R}$ and $(P, \phi), \phi : P \to P$

Interval $[j, j+1], j = 1, ..., n-1$, is called *P-basic*

P-linear map (or relative map) denoted as f_P is a continuous map of the $convex(P) = [1, n]$ into itself such that

$$\begin{cases} f_P|_P = \phi \\ f_P|_J \text{ is affine for any interval } J \subset convex(P) \text{ for which } J \cap P = \emptyset \end{cases}$$

Topological entropy

Topological entropy of relative maps of permutations of P

Matrix $A = A(P, \phi)$ is the $(n-1) \times (n-1)$ matrix defined as follows:

$$A_{JK} = \begin{cases} 1 \ \text{ if } K \subset f_P(J) \\ \\ 0 \ \text{ otherwise} \end{cases}$$

The entropy is defined as $h((P, \phi)) = \log r(A)$, where $r(A)$ is the spectral radius of A.

Classification of cyclic permutations: let (P, ϕ) possess a unique fixed point c

P_L, P_R denote the *left* and *right* part of P with respect to c respectively

$$P_G = \{x \in P : (x - c)(\phi(x) - c) > 0\}, \ P_B = P \backslash P_G$$

A switch is a P-basic interval with endpoints from different sets P_L and P_R

A height of point $x \in P$ denoted by $H(x)$ is the number of switches between x and $\phi^2(x)$

Cyclic permutation (P, ϕ) is *green* if $P_R \subset P_B$, ϕ is increasing on $P_G \neq \emptyset$ and decreasing on P_B

Complexity denoted by $C(P)$ is the maximum of heights of the points from $P_L \cap P_B$.

2.1 Theorem [1] *Denoting* $G_N = \{(P,\phi) : C(P) \le 2N\}$ *we have that for* $(P,\phi) \in G_N$

$$h(P,\phi) \in [\frac{1}{2}\log C(P), \log \alpha(N)],$$

and

$$\sup \{h(P,\phi) \in G_N\} = \log \alpha(N),$$

where $\alpha(N)$ *is a positive root of the polynomial equation*

$$\frac{1}{\alpha^2}\left(\frac{\alpha+1}{\alpha-1}\right)^N = N^N \frac{\sqrt{1+N^2}-N}{(1+\sqrt{1+N^2})^N},$$

and simultaneously, it is the least positive value for which the difference equation

$$\xi_{k+N+1} = \frac{-\alpha+1}{\alpha^3+\alpha^2}\xi_k - \frac{1}{\alpha^2}\xi_{k+1} + \frac{\alpha-1}{\alpha+1}\xi_{k+N} + 2,\ k = 0, 1, ...$$

possesses a strictly monotone solution $\{\xi_k\}_{k\ge 0}$

A nonempty closed set $\mathcal{K} \subset \mathcal{E}$ where $\mathcal{E}$ is a Banach space with norm $\|.\|$ is called a *cone* if it satisfies

(i) $\mathcal{K}+\mathcal{K} \subset \mathcal{K}$, (ii) $a\mathcal{K} \subset \mathcal{K}$ for $a \in \mathcal{R}_+ = [0, +\infty)$ (iii) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$

If $\mathcal{K}$ is a cone in $\mathcal{E}$ we write $x \leq y, x, y \in \mathcal{E}$ or equivalently, $y \geq x$ whenever $(y - x) \in \mathcal{K}$

A cone is called

(iv) *generating* $\mathcal{G}$ if $\mathcal{G} = \mathcal{K} - \mathcal{K}$ is a (norm)-closed subspace of $\mathcal{E}$.

(v) *normal* if there is a constant δ such that $\|x\| \leq \delta\|y\|$ whenever $0 \leq x \leq y$.

A bounded linear operator T mapping $\mathcal{E}$ into $\mathcal{E}$ is called $\mathcal{K}$-positive if $T\mathcal{K} \subset \mathcal{K}$.

$\mathcal{F} = \mathcal{E} + (i\mathcal{E}),\ i^2 = -1$ complex extension of $\mathcal{E}$

$T : \mathcal{E} \longrightarrow \mathcal{E} \quad \tilde{T} = T + iT$ complex extension of T

$\sigma(\tilde{T})$ spectrum of T, $r(\tilde{T})$ spectral radius of T

Let μ be an isolated singularity of the resolvent operator $R(\lambda, \tilde{T}) = (\lambda I - \tilde{T})^{-1}$ of $\tilde{T}$

$$R(\lambda, \tilde{T}) = \sum_{k=0}^{\infty} A_k(\mu)(\lambda - \mu)^k + \sum_{k=1}^{\infty} B_k(\mu)(\lambda - \mu)^{-k}, \tag{3.1}$$

where $A_{k-1}(\mu)$ and $B_k(\mu)$, $k = 1, 2, ...$, belong to $\mathbf{B}(\mathcal{F})$. In particular,

$$B_1(\mu) = \frac{1}{2\pi i} \int_{\{\lambda:\ |\lambda - \mu| = \rho_0\}} (\lambda I - \tilde{T})^{-1} \mathrm{d}\lambda, \tag{3.2}$$

where $\{\lambda\colon\ |\lambda - \mu| \leq \rho_0\} \cap \sigma(\tilde{T}) = \{\mu\}$ and

$$B_1^2(\mu) = B_1(\mu). \tag{3.3}$$

Furthermore,

(3.4) $B_{k+1}(\mu) = (\tilde{T} - \mu I)B_k(\mu) = B_k(\mu)(\tilde{T} - \mu I),\ k = 1, 2, ...$

If there is a positive integer $q = q(\mu)$ such that

$$B_q(\mu) \neq \Theta, \text{ and } B_k(\mu) = \Theta \text{ for } k > q(\mu),$$

then μ is called *a pole of the resolvent operator* and q is its *multiplicity*.

The following statement will be useful when proving our main results.

3.1 Theorem $[12, Theorem 5.8 - A]$ *For a closed linear operator* $\tilde{T}\mathcal{F} \to \mathcal{F}$ *let* $\mu \in \mathcal{C}$ *be a pole of the resolvent operator* $R(\lambda, \tilde{T})$ *with a multiplicity* $q(\mu)$*. Then* μ *is an eigenvalue of the operator* $\tilde{T}$ *and the range of the projection* $B_1(\mu)$ *equals to the kernel of the operator* $(\mu I - \tilde{T})^{q(\mu)}$*.*

Cone $K \subset \mathcal{E}$ is essential for a bounded linear operator $T : \mathcal{E} \to \mathcal{E}$ if T is $\mathcal{K}$-positive and for $\mathcal{L} = \overline{\mathcal{K} - \mathcal{K}}$, $r(T|_{\mathcal{L}}) > 0$.

For a sequence $\mathbf{x} = \{x_k\}_{k \geq 0} \subset \mathcal{E}$ let $\mathcal{J} = \{x_{k+1} - x_k\colon\ k \geq 0\}$ and

$$\mathcal{K}(\mathbf{x}) = \overline{\{\sum_{i=1}^{n} \alpha_i v_i\colon\ n \in \mathcal{N},\ \alpha_i \in \mathcal{R}_+,\ v_i \in \mathcal{J}\}}. \tag{3.5}$$

Let $T : \mathcal{E} \to \mathcal{E}$ and $x_0, w \in \mathcal{E}$. The sequence $\mathbf{x} = \{w + T^k x_0\}_{k \geq 0}$ is said to be strictly monotone if the set $\mathcal{K}(\mathbf{x})$ is a cone essential for the operator T.

3.2 Problem *Given an operator* $T \in \mathbf{B}(\mathcal{E})$. *To find a vector* $x_0 \in \mathcal{E}$ *such that the sequence* $\{x_k\}_{k \geq 0}$ *given by*

$$x_{k+1} = Tx_k, \tag{3.6}$$

is strictly monotone.

3.3 Theorem Krein-Schaefer [4, Theorem 9.2] *Suppose that a completely continuous* $\mathcal{K}$*-positive operator* T *satisfies* $r(T) > 0$ *and that* $\overline{\mathcal{K} - \mathcal{K}} = \mathcal{E}$. *Then* $r(T)$ *is an eigenvalue of* T *with corresponding eigenvector in* $\mathcal{K}$.

4.1 Theorem *Let $T \in \mathbf{B}(\mathcal{E})$ be completely continuous. The following two statements are equivalent:*

(i) There is an x_0 such that the sequence $\{T^k x_0\}_{k \geq 0}$ is strictly monotone.

(ii) The spectrum $\sigma(T)$ of T contains an eigenvalue $\mu > 0$ such that either $\mu \neq 1$ or $\mu = 1$ is a pole of the resolvent operator with multiplicity $q(1) \geq 2$.

Proof

(i)$\Rightarrow$(ii). Assuming a strictly monotone sequence $\{T^k x_0\}_{k\geq 0}$ we get that the set $\mathcal{K} = \mathcal{K}(\mathbf{x})$ defined in (3.5) is an essential cone for the operator T. Set $\mathcal{L} = \overline{\mathcal{K} - \mathcal{K}}$. Theorem 3.3 implies that $r(T|_{\mathcal{L}}) \in \sigma(T|_{\mathcal{L}}) \subset \sigma(T)$ and since $r(T|_{\mathcal{L}}) > 0$ it is an eigenvalue of T with corresponding eigenvector u in $\mathcal{K}$.

Let $r(T|_{\mathcal{L}}) = 1$ be the only positive element in $\sigma(T)$ and let it be a pole of the resolvent operator with multiplicity $q(1) = 1$. Then the operators $B_k(1)$ from the Laurent series (3.1) satisfy

$$B_1(1) \neq \Theta,\ B_k(1) = \Theta,\ k = 2, 3, \ldots. \tag{4.1}$$

Moreover, by Theorem 3.1 and (3.3), for some $x \in \mathcal{L}$,

$$u = B_1(1)x = B_1^2(1)x, \text{ hence } B_1(1)u = u.$$

Since $u \in \mathcal{K}$ and $B_1(1)$ is bounded, a nonnegative integer k must exist such that $x_{k+1} - x_k$ for which

$$B_1(1)(x_{k+1} - x_k) \neq \theta. \tag{4.2}$$

At the same time by (3.6),(3.4) and (4.1)

$$B_1(1)(x_{k+1} - x_k) = B_1(1)(T - I)x_k = B_2(1)x_k = \theta, \tag{4.3}$$

a contradiction. Thus, $q(1) \geq 2$.

(ii)$\Rightarrow$(i). First, let $\mu \in \sigma(T)$, $\mu > 0$ and $\mu \neq 1$. Choose an eigenvector $x_0 \in \mathcal{E}$ corresponding to μ and consider $\mathbf{x} = \{x_k\}_{k \geq 0}$ and $\mathcal{K} = \mathcal{K}(\mathbf{x})$. By (3.5), either $\mathcal{K} = \{\alpha x_0\colon\ \alpha \in \mathcal{R}_+\}$ for $\mu > 1$ or $\mathcal{K} = \{-\alpha x_0\colon\ \alpha \in \mathcal{R}_+\}$ if $\mu \in (0, 1)$. In any case $\mathcal{K} \neq \theta$ and clearly it is an essential cone for the operator T. This proves the first part of our conclusion.

Second, let the spectrum $\sigma(T)$ contain value 1 as a pole of the resolvent operator (3.1) with multiplicity $q(1) \geq 2$. Moreover, let $y_0 \in \mathcal{E}$ be such that $B_s(1)y_0 \neq \theta$ and $B_{s+1}(1)y_0 = \theta$ with an appropriate $1 < s \leq q(1)$. Setting $x_0 = B_s(1)y_0 + B_{s-1}(1)y_0$ and using (3.4) repeatedly we get for each $k \geq 0$

$$x_k = T^k x_0 = (k+1)B_s(1)y_0 + B_{s-1}(1)y_0, \tag{4.4}$$

hence the sequence $\mathbf{x} = \{x_k\}_{k\geq 0}$ satisfies

$$\mathcal{K} = \mathcal{K}(\mathbf{x}) = \{\alpha B_s(1)y_0\colon\ \alpha \in \mathcal{R}_+\}.$$

Since again by (3.4), $TB_s(1)y_0 = B_s(1)y_0$, we conclude that $\mathcal{K} \neq \{\theta\}$ is a cone essential for the operator T, i.e., the sequence $\mathbf{x} = \{x_k = T^k x_0\}_{k\geq 0}$ is strictly monotone.

As an application of our Theorem 4.1 we present the proof of Theorem 2.1.

Proof of Theorem 2.1.

Let $\mathcal{E} = \mathcal{R}^N$,

$$T = \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 1 \\ a_1 & a_2 & a_3 & . & . & . & a_{N-1} & a_N \end{pmatrix}, \tag{4.5}$$

where $a_1, ..., a_N$ come from (1.1), and let

$$X_k = \begin{pmatrix} x_k \\ . \\ . \\ . \\ x_{k+N-1} \end{pmatrix}.$$

The sufficiency directly follows from the formula

$$x_k = \sum_{j=1}^{n} \sum_{\ell=0}^{m_j-1} c_{j,\ell}\, k^\ell \mu_j^k, \tag{4.6}$$

(the complex coefficients $c_{j,\ell}$ are uniquely determined by an initial condition for the values $x_0, \ldots, x_{N-1}$) where $\mu_1, \ldots, \mu_n$ are all distinct roots of the characteristic polynomial and m_j is a multiplicity of μ_j. We thus have

$$X_k = T^k X_0 = \sum_{j=1}^{n} \sum_{\ell=1}^{m_j} \mu_j^k k^{\ell-1} B_{j,\ell} X_0, \tag{4.7}$$

where the operators $B_\ell, \ell = 1, ..., m_j$ denotes appropriate analogs of the operators introduced in (3.1) and (3.4)

Then, T is completely continuous and (1.1) is equivalent to

$$X_{k+1} = TX_k, \ \ k = 0, 1, \ldots . \tag{4.8}$$

Without loss of generality assume that the sequence $\{x_k\}$ is strictly increasing (the proof for a strictly decreasing sequence is analogous). Let $\mathcal{K} = \mathcal{K}(\mathbf{X})$ be defined as in (3.5). Clearly, $\mathcal{K}$ is a subcone of the cone $\mathcal{R}_+^N \subset \mathcal{R}^N$ and $\mathcal{L} = \mathcal{K} - \mathcal{K}$ is a closed subspace of $\mathcal{R}^N$. Let $k \geq 1$ be fixed. Then

$$\zeta = \min \left\{ \frac{x_{k+j+1} - x_{k+j}}{x_{k+j} - x_{k+j-1}} : j = 0, ..., N-1 \right\} > 0. \tag{4.9}$$

We also have the relations

$$\begin{aligned} Y_{k+1} = X_{k+1} - X_k &= T\left(X_k - X_{k-1}\right)) \\ &\geq \zeta\left(X_k - X_{k-1}\right) \\ &= \zeta Y_k. \end{aligned}$$

4.2 Proposition [4, Lemma 9.1] *Suppose that $T : \mathcal{E} \to \mathcal{E}$ is a $\mathcal{K}$-positive operator bounded linear operator such that some element $u \notin -\mathcal{K}$ satisfies*

$$Tu \geq \nu u, \tag{4.10}$$

where $\nu \geq 0$. Then $r(T) \geq \nu$.

Since T is $\mathcal{K}$-positive, we deduce from Proposition 4.2 that $r(T|_{\mathcal{L}}) \geq \zeta$ hence by Theorem 3.3, $r(T|_{\mathcal{L}}) > 0$ is an eigenvalue of $T|_{\mathcal{L}}$ with corresponding eigenvector $U \in \mathcal{K}$. It shows that the cone $\mathcal{K}$ is essential for T, i.e., the sequence $\{X_k\}_{k\geq 0}$ is strictly monotone due to definition and the conclusion follows from Theorem 4.1.

5

The similarity of the theory of ordinary linear differential equations and linear difference equations with constant coefficients suggests considering the following problem. In the sequel the symbol $x^{(j)}(t)$ denotes the jth derivatives of a function $x(t)$.

5.1 *Problem Let* $N \geq 1$, $b_1, \ldots, b_N \in \mathcal{R}$. *Find* $y = y(t)$ *such that*

$$y^{(N)}(t) = b_1 y(t) + b_2 y^{(1)}(t) + \ldots + b_N y^{(N-1)}(t), \ t \in \mathcal{R} \tag{5.1}$$

and satisfying either

$$y(s) < y(t) \text{ for all } t > s \geq 0 \tag{5.2}$$

or

$$y(s) > y(t) \text{ for all } t > s \geq 0. \tag{5.3}$$

5.2 Theorem *The following statements are equivalent.*

(i) There exists a solution $y = y(t)$ to (5.1) such that (5.2) or (5.3) holds.

(ii) The characteristic polynomial to (5.1) possesses either a real root $\lambda \neq 0$ or $\lambda = 0$ is a root whose multiplicity is at least two.

Proof Denote $\lambda_1, \ldots, \lambda_n$ all distinct roots of characteristic polynomial of (5.1). Obviously, $n \leq N$.

Let

$$Y(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \cdot \\ \cdot \\ \cdot \\ y^{(N-1)}(t) \end{pmatrix}, t \geq 0, \quad Y(0) = \begin{pmatrix} y(0) \\ y'(0) \\ \cdot \\ \cdot \\ \cdot \\ y^{(N-1)}(0) \end{pmatrix},$$

where $Y(0)$ is the vector of initial values, From the well known fact

$$Y(t) = \exp\{Tt\}Y(0) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \exp\{\lambda_j t\} \frac{t^{k-1}}{(k-1)!} B_{j,k} Y(0)$$

we derive that for

$$Z_\ell = Y\left(\frac{\ell}{q}\right) = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \exp\left\{\lambda_j \frac{\ell}{q}\right\} \frac{\left(\frac{\ell}{q}\right)^{k-1}}{(k-1)!} B_{j,k} Y(0)$$

where q is any positive integer and $l = 0, 1., ...,$
we have

$$Z_\ell = A^\ell Z_0 = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \mu_j^\ell \left(\frac{\ell}{q}\right)^{k-1} \frac{1}{(k-1)!} B_{jk} Y(0)$$

where

$$A = \exp\{\frac{1}{q} Tt\},\ \lambda_j = q \log \mu_j.$$

According to Answer 2.1 the required assertion follows.

2.1 Answer Let $N \geq 1$ and

$$x_{k+N} = a_1 x_k + a_2 x_{k+1} + ... + a_N x_{k+N-1}, \;\; k = 0, 1, ..., \tag{5.4}$$

where $a_1, ..., a_N$ are arbitrary real numbers. A strictly monotone solution of the given difference equation (5.4) exists if and only if the characteristic polynomial p of (5.4) possesses either a positive root $\mu \neq 1$ or $\mu = 1$ is a root of p whose multiplicity is at least two.

6 Monotone semigroups of operators

The previous generalizations can be extended to some cases of (C_0)-semigroups of linear operators and in particular linear PDE's. For the sake of completeness we outline possible approach here. For more detailed information related to this paragraph - see [3].

By a (C_0)-semigroup we mean a one-parameter system $\mathcal{T} = \{T(t)\}_{t \geq 0}$ of operators from the space of bounded linear operators $\mathbf{B}(\mathcal{E})$ with the property that

$$T(s+t)x = T(s)T(t)x,\ T(0) = I \tag{6.1}$$

for all $s, t \in \mathcal{R}_+$ and all $x \in \mathcal{E}$. We assume that $T(t)$ is strongly continuous for $t \geq 0$, i.e. for each $x \in \mathcal{E}$

$$\lim_{h \to 0} T(t+h)x = T(t)x. \tag{6.2}$$

The infinitesimal generator B of $\mathcal{T} = \{T(t)\}_{t\geq 0}$ is a closed (generally unbounded) operator defined by

$$Bx = \lim_{h\to 0+} \frac{1}{h}(T(h) - I)x \tag{6.3}$$

whenever the limit exists. The domain $\mathcal{D}(B)$ is dense in $\mathcal{E}$. For each $x \in \mathcal{D}(B)$ ([3, Theorem 10.3.3]),

$$\frac{d}{dt}T(t)x = BT(t)x = T(t)Bx \tag{6.4}$$

and for $x \in \mathcal{D}(B^n)$, $n \in \mathcal{N}$, it is possible to represent $T(t)x$ by means of an "exponential formula" ([3, p. 354])

$$T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}B^k x + \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}T(\tau)B^n x\, d\tau. \tag{6.5}$$

The point and residual spectrum of $\mathcal{T} = \{T(t)\}_{t\geq 0}$ come from those of its generator B.

6.1 Theorem *[3, Theorems 16.7.1-3] (i) $P\sigma[T(t)] = exp[tP\sigma(B)]$, plus, possibly, the point $\lambda = 0$. If $\mu \in P\sigma[T(t)]$ for some fixed $t > 0$ where $\mu \neq 0$ and if $\{\alpha_n\}$ is the set of roots of $exp(t\alpha) = \mu$ then at least one of the points α_n lies in $P\sigma(B)$. (ii) If $\mu \in R\sigma[T(t)]$ for some fixed $t > 0$ where $\mu \neq 0$, then at least one of the solutions of $exp(t\alpha) = \mu$ lies in $R\sigma(B)$ and none can lie in $P\sigma(B)$. (iii) $exp[tC\sigma(B)] \subset C\sigma[T(t)]$.*

A cone $\mathcal{K} \subset \mathcal{E}$ is called essential for a (C_0)-semigroup $\mathcal{T} = \{T(t)\}_{t\geq 0}$ if each $T(t)$ is $\mathcal{K}$-positive and for $\mathcal{L} = \overline{\mathcal{K} - \mathcal{K}}$, $\mathcal{T}_{|\mathcal{L}}$ is generated by an operator B such that $\sigma(B) \neq \emptyset$.

For a collection $\mathbf{x} = \{x(t)\}_{0\leq t<\tau} \subset \mathcal{E}$, $0 < \tau \leq \infty$, let $\mathcal{J}(\mathbf{x}) = \{\frac{d}{dt}x(t)\colon\ 0 \leq t < \tau\}$ and let $\mathcal{K}(\mathbf{x})$ be given by (3.5).

Analogously as in the previous we define the following.

6.2 Definition *Let* $\mathcal{T} = \{T(t)\}_{t\geq 0} \subset \mathbf{B}(\mathcal{E})$ *be a* (C_0)*-semigroup with an infinitesimal generator* B, $x_0 \in \mathcal{D}(B)$, $w \in \mathcal{E}$. *The collection* $\mathbf{x} = \{w + T(t)x_0\}_{t\geq 0}$ *is said to be strictly monotone if the set* $\mathcal{K}(\mathbf{x})$ *is a normal generating cone essential for the semigroup* $\mathcal{T}$.

6.3 Problem *Given a closed linear operator* B *with the domain* $\mathcal{D}(B) \subset \mathcal{E}$ *and generating a* (C_0)*-semigroup* $\{T(t)\}_{t\geq 0} \subset \mathbf{B}(\mathcal{E})$. *To find vector* $x_0 \in \mathcal{D}(B)$ *such that the collection* $\{x(t) = T(t)x_0\}_{t\geq 0}$ *satisfying*

$$\frac{d}{dt}x(t) = Bx(t),\ x(0) = x_0,\ t \geq 0 \tag{6.6}$$

is strictly monotone.

For the reader's convenience we present the Krein-Schaefer Theorem in a form which is applied in our contribution.

6.4 Theorem [5], [10, Proposition 4.1], [11] *Let $\mathcal{E}$ be a Banach space generated by a normal cone $\mathcal{K}$. If $T \in \mathbf{B}(\mathcal{E})$ is $\mathcal{K}$-positive then the spectral radius $r(T)$ belongs to the spectrum $\sigma(T)$.*

6.5 Definition *Let* $\mathcal{T} = \{T(t)\}_{t\geq 0} \subset \mathbf{B}(\mathcal{E})$ *be a* (C_0)*-semigroup with an infinitesimal generator* B*. We say that the infinitesimal generator* B *of a* (C_0)*-semigroup* $\mathcal{T} = \{T(t)\}_{t\geq 0}$ *is admissible if*

- $C\sigma(T(t)) = \emptyset$ *for each* $t > 0$;
- $\sigma(B) \cap \mathcal{R} \subset P\sigma(B)$;
- $t(\sigma(B) \setminus \mathcal{R}) \cap (\mathcal{R} \times \{\pi n i\}_n \in \mathcal{Z}) = \emptyset$ *for some* $t > 0$;
- *and if* $\sigma(B) \cap \mathcal{R} = \{0\}$ *then* 0 *is a pole of the resolvent operator of* B.

6.6 Theorem *Concerning Problem 6.3 assume that the operator B is admissible. Then the following two statements are equivalent.*

(i) There is an $x_0 \in \mathcal{D}(B)$ such that the corresponding solution $\{x(t)\}_{t\geq 0}$ of (6.6) is strictly monotone.

(ii) The spectrum $\sigma(B)$ contains a real eigenvalue μ. This eigenvalue is either nonzero or 0 is a pole of the resolvent operator whose multiplicity is at least 2.

Proof (i)⇒(ii). Let $\{x(t) = T(t)x_0\}_{t\geq 0}$, $x_0 \in \mathcal{D}(B)$, be a strictly monotone solution to (6.6). By our definition, every element of the semigroup $\mathcal{T}$ is $\mathcal{K}$-positive, where $\mathcal{K} = \mathcal{K}(\mathbf{x})$ is a normal generating cone due to Definition 6.2. Let $\mathcal{L} = \mathcal{K} - \mathcal{K}$. Applying Theorem 6.4 we get that for every $t \geq 0$, the spectral radius $r_t = r(T(t)|_{\mathcal{L}})$ belongs to the spectrum $\sigma(T(t)|_{\mathcal{L}})$. Since B is admissible and $\mathcal{K}$ is essential for $\mathcal{T}$, Theorem 6.1 implies

$$r_t = e^{tz_t} > 0 \text{ for some } z_t \in P\sigma(B) \text{and each } t \geq 0. \tag{6.7}$$

If it were $\sigma(B) \cap \mathcal{R} = \emptyset$ then, again by the admissibility of B, there would exist a positive t for which $(\mathcal{R} \times \{\pi n i\}_n \in \mathcal{Z}) \cap t\sigma(B) = \emptyset$, what contradicts (6.7). Thus, the spectrum $\sigma(B)$ of B has to contain a real eigenvalue μ.

To finish the proof of this part, let us assume that $\mu = 0$ is the only real element in $\sigma(B)$. By our assumption on B, then 0 is a pole (an isolated singularity) of the resolvent operator of B. We can consider the Laurent expansion of the resolvent operator about 0 and the operator

$B_1(0)$ given by (3.2). Let the 0 be a pole of multiplicity $q(0) = 1$.

Using the expression (3.1) and the operators $B_k(0)$ from (3.4) for the operator B, we get

$$B_k(0) = \Theta,\ k = 2, 3, ...; \tag{6.8}$$

moreover, Theorem 3.1 and (3.3) imply that for some $y, x \in \mathcal{L}$, $x \neq 0$ and $u, v \in \mathcal{K}$,

$$Bx = 0,\ B_1(0)y = x,\ B_1^2(0)y = B_1(0)x = x,\ x = u - v,$$

hence for some $w \in \{u, v\}$, $B_1(0)w \neq 0$. On the one hand, since $B_1(0)$ is bounded and $\mathrm{Lin}\{\mathcal{J}(\mathbf{x})\}$ is dense in $\mathcal{K}$, there has to exist an element $w_0 = \frac{d}{dt}x(t)|_{t=t_0} \in \mathcal{J}(\mathbf{x})$ such that

$$B_1(0)w_0 \neq 0. \tag{6.9}$$

On the other hand, from (6.4) and (3.4) we get

$$
\begin{aligned}
B_1(0)w_0 &= B_1(0)T(t_0)Bx_0 \\
&= B_1(0)BT(t_0)x_0 \qquad (6.10)\\
&= B_2(0)T(t_0)x_0 = 0,
\end{aligned}
$$

a contradiction. Thus, $q(0) \geq 2$.

(ii)$\Rightarrow$(i). For B admissible, let $0 \neq \mu \in P\sigma(B) \cap \mathcal{R}$ and for $0 \neq u \in \mathcal{D}(B)$, $Bu = \mu u$. Applying formula (6.5) we obtain $x(t) = T(t)u = e^{t\mu}u$, hence

$$
\mathcal{J}(\mathbf{x}) = \{\frac{d}{dt}x(t) = \mu e^{t\mu}u\colon\ t \geq 0\} \qquad (6.11)
$$

$$
\mathcal{K}(\mathbf{x}) = \{\alpha\mu u\colon\ \alpha \in \mathcal{R}_+\}.
$$

The reader can easily verify that $\mathcal{K}(\mathbf{x})$ is a normal generating cone essential for the semigroup $\mathcal{T}$ reduced to $\{\alpha u\colon\ \alpha \in \mathcal{R}\}$. If $\mu = 0$ and $q = q(0) > 1$, we can choose $y_0 \in \mathcal{E}$ and $1 < s \leq q$ such that $B_s y_0 \neq 0$ and $B_{s+1} y_0 = 0$. Put $x_0 = B_s y_0 + B_{s-1} y_0$. Using (6.5) we get

$$x(t) = T(t)x_0 = (t+1)B_s y_0 + B_{s-1} y_0 \tag{6.12}$$

hence

$$\mathcal{J}(\mathbf{x}) = \left\{\frac{d}{dt}x(t) = B_s y_0\right\},\ \mathcal{K}(\mathbf{x}) = \{\alpha B_s y_0\colon\ \alpha \in \mathcal{R}_+\} \tag{6.13}$$

and $\mathcal{K}(\mathbf{x})$ is a normal generating cone that is essential for the semigroup $\mathcal{T}$ reduced to $\{\alpha B_s y_0\colon\ \alpha \in \mathcal{R}\}$.

6.7 Remark *Consider a nonhomogeneous version of (6.6), i.e., for* $b \neq \theta$ *and a given closed linear operator* B *the equation*

$$\frac{d}{dt}x(t) = Bx(t) + b,\ x(0) = x_0,\ t \geq 0; \tag{6.14}$$

let us assume that there exists an element $w \in \mathcal{D}(B)$ *such that*

$$Bw = -b. \tag{6.15}$$

We can easily verify that $\{y(t) = w + x(t)\}_{t\geq 0}$*, with* $\{x(t)\}_{t\geq 0}$ *being a solution of the appropriate homogeneous equation (6.6), is a solution of equation (6.14). Moreover, the condition (ii) in Theorem 6.6 is necessary and sufficient for a solution* $\{y(t)\}_{t\geq 0}$ *of (6.14) to be strictly monotone due to Definition 6.2.*

7 Concluding remarks

Let us comment on the necessary and sufficient conditions guaranteeing existence of strictly increasing solutions to linear operators we studied in the previous sections.

The condition concerned with multiplicity of value one as a point of the spectrum of the operator under consideration indicate that operators satisfying this condition must be in a sense "strange". In other words, most of the operators appearing as governing operators in mathematical modeling in Science, Economics, Engineering etc. cannot be expected among them.

A rather broad class of operators for which the existence of strictly increasing solutions cannot be guaranteed is a subclass of operators T whose some function $f(T)$ leaves some generating normal cone $\mathcal{K}$ invariant and is irreducible (see [5], [6]).

As example, we can take $N \times N$ matrix $T = I - B : B$ is stochastic irreducible and such that its spectrum $\sigma(T) \subset \{1\} \cup \{\Re\lambda < 0\} \cup \{\lambda : \lambda = \mu + i\nu, \mu, \nu$ real $\nu \neq 0\}$ [8], [9]. A very special subclass of the class just described is formed by irreducible p-cyclic stochastic operators.

References

[1] Bobok J. *On entropy of patterns given by interval maps*. Fundamenta Mathematicae **162**(1999), 1-36.

[2] Bobok J., Kuchta M. *X-minimal patterns and generalization of Sharkovskii's Theorem*. Fundamenta Mathematicae **156**(1998), 33-66.

[3] Hille E., Phillips R.S.*Functional Analysis and Semigroups* Amer. Math. Socitey Coll. Publ. Vol XXXI, Third printing of Revised Edition, Providence, Rhode Island 1968.

[4] Krasnosel̆skij M.A., Lifshits Je.A., Sobolev A.V. *Positive Linear Systems - The Method of Positive Operators*. Sigma Series in Applied Mathematics 5, Heldermann Verlag Berlin, 1989.

[5] Krein M. G., Rutman M.A. *Linear operators leaving invariant a cone in a Banach space*. Uspekhi mat. nauk **III**, Nr. 1, 3 - 95 (1948). (In Russian.)

[6] Marek I. *Frobenius theory of positive operators. Comparison theorems and applications*. SIAM J. Appl. Math. **19** (1970), 607-628.

[7] Marek I. *Some spectral properties of Radon-Nikolskii operators and their generalizations*. Comment. Math. Univ. Carol. **3**, Nr.1, 20-30 (1962)

[8] Papáček Š. (2005) *Photobioreactors for Cultivation of Microalgae Under Strong Irradiances* Modelling: Simulation and Design. Ph.D. Thesis, Technical University Liberec

[9] Papáček Š, Čelikovský S., Štys D., Ruiz-León J. (2007) Bilinear system as a modelling framework for analysis of microalgal growth. Kybernetika 43: 1-20

[10] Schaefer H.H. *Banach Lattices and Positive Operators*. Springer-Verlag Berlin-Heidelberg-New York, 1974.

[11] Schaefer H.H. *On the singularities of an analytic function with values in a banach space*. Archiv der Mathematik Band **129** (1960), 323-329.

[12] Taylor A.E. *Introduction to Functional Analysis*. Wiley Publ. New York 1958.