$(0, 1)$-Matrices and Nonnegative Eigenvalues

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Joint Work with Steve Kirkland
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Dedicated to my friend/co-worker of many years

Last ILAS Meeting

Focus

- (0,1)-matrices
- Matrices each of whose eigenvalues is a nonnegative real number.

Examples

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Eigenvalues of A_4: $0, 0, 2 \pm \sqrt{2}$ (all nonnegative).

Eigenvalues of A_5: $0, 0, 1, 2 \pm \sqrt{3}$ (all nonnegative).

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Examples cont.

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Eigenvalues of A_6: $0, 0, 0, 1, 1, 4$ (all nonnegative).

Eigenvalues of A_7: $0, 0, 0, 1.3194 \pm 0.49781i, 0.1185, 4.2426$ (not even all real).

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Examples cont.

Each of these examples is obtained from the **full lower $(0,1)$-Hessenberg** H_n of order n by replacing the **0** in the upper right corner with a 1. For instance,

$$H_6 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \mathbf{0} \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

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Theorem of McKay, Oggier, Royle, Sloane, Wanless, Wilf (2004)

Theorem: Let A be a $(0,1)$-matrix of order n **all** of whose eigenvalues are **positive**. Then actually all eigenvalues equal 1, and there is a permutation matrix P such that

$$PAP^t = I_n + B$$

where B is a lower triangular matrix with 0s on and above the main diagonal.

The theorem asserts that **the digraph of A does not have any cycles of length > 1.**

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Theorem cont.

Recall that a matrix A of order n is **irreducible** provided that there does **not** exist a permutation matrix P such that

$$PAP^t = \left[\begin{array}{c|c} A_1 & O_{r,n-r} \\ \hline A_{21} & A_2 \end{array}\right].$$

This is equivalent to the digraph of A being strongly connected.

Corollary to Theorem: An **irreducible** $(0,1)$-matrix A of order $\mathbf{n \geq 2}$ with all eigenvalues **nonnegative** has 0 as an eigenvalue and hence is singular.

(Otherwise, by the theorem A is triangular and hence reducible.)

Theorem generalized

Theorem: Let A be a $(0, 1)$-matrix of order n with trace at most r and with r positive eigenvalues and $n - r$ zero eigenvalues. Then there is a permutation matrix P such that $PAP^T = D + B$ where B is a $(0, 1)$-matrix with 0s on and above the main diagonal and D is a $(0, 1)$-diagonal matrix with r 1s. **In particular, A has r eigenvalues equal to 1, $n - r$ eigenvalues equal to 0, and the trace of A equals r.**

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Outline of Proof

Let $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n$ be the eigenvalues of A. The AM/GM inequality gives

$$1 \geq \frac{\text{trace}(A)}{r} = \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_r}{r} \geq (\lambda_1 \lambda_2 \cdots \lambda_r)^{1/r}.$$

The sum α_r of the determinants of the principal submatrices of order r of A (necessarily an integer) equals the sum of the products of the eigenvalues of A taken r at a time and so equals $\lambda_1 \lambda_2 \cdots \lambda_r$ and is positive. Thus $\alpha_r = \lambda_1 \lambda_2 \cdots \lambda_r \geq 1$. So

$$1 \geq \frac{\text{trace}(A)}{r} = \frac{\lambda_1 + \lambda_2 \cdots + \lambda_r}{r} \geq (\lambda_1 \lambda_2 \cdots \lambda_r)^{1/r} \geq 1.$$

So equality throughout, and $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 1$. Thus A has r eigenvalues equal to 1, and $n - r$ eigenvalues equal to 0, and the trace of A equals r.

Outline of Proof cont.

The Perron-Frobenius theory implies that A has r irreducible components $A_1, A_2, \ldots, A_r$ each of which has spectral radius 1, and all other eigenvalues equal to 0; the remaining irreducible components, if any, are zero matrices of order 1.

Since each A_i is irreducible, each A_i has at least one 1 in each row and column. The Perron-Frobenius theory implies each A_i is a permutation matrix corresponding to a permutation cycle. Since the eigenvalues of A_i are one 1 and then all 0s, each A_i has order 1. Thus A has r 1s and $n - r$ 0s on the main diagonal, and all 0s above the main diagonal.

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Example

The irreducible matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

of order 4 has trace equal to 4 and nonnegative eigenvalues $0, 1, (3 \pm \sqrt{5})/2$ of which 3 are positive. Since this matrix is irreducible, it cannot be simultaneously permuted to a triangular matrix. Thus the bound on the trace in the theorem is important.

How to get a handle on matrices with all eigenvalues nonnegative?

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How to get a handle on matrices with all eigenvalues nonnegative?

TNN Matrices

A **TNN matrix** is a (rectangular) matrix in which the determinants of all square submatrices (in particular, all entries) are **nonnegative**.

Example: $A = \begin{bmatrix} 4 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

Basic Property of square TNN matrices: All eigenvalues are nonnegative.

This property does not characterize square TNN matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

has eigenvalues $0, 2, 2$ but A is not TNN (submatrix of order 2 in upper right corner), nor can the rows and columns be simultaneously permuted to a TNN matrix..

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Question

The property for a (0,1)-matrix to be TNN (all submatrices have nonnegative determinants) is easier to handle than the weaker property that all eigenvalues are nonnegative.

What are the special properties TNN (0,1)-matrices?

de Boor - Pinkus Theorem 1982

In a TNN matrix with no zero rows or columns, the nonzero entries in each row and in each column occur consecutively. Moreover, the first and last nonzero entries in a row (resp. column) are not to the left of the first and last nonzero entries, respectively, in any preceding row (resp. column).

Thus if $*$ denotes a nonzero entry, the such a TNN matrix has the form

$$\begin{bmatrix} * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \end{bmatrix}.$$

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$(0,1)$ Case of de Boor - Pinkus Theorem 1982

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The 1s have a **double staircase pattern**.

So in investigating $(0,1)$-TNN matrices, it can be assumed that the 1s have this double staircase pattern.

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Proof of Double Staircase pattern

A cannot have any submatrix of order 2 equal to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Consider any 0 in A:

$$A = \left[\begin{array}{c|c|c} & \alpha & \\ \hline \beta & 0 & \gamma \\ \hline & \delta & \end{array}\right]$$

. Not both α and β (γ and δ) can contain a 1. So α is a zero column or β is a zero row, and γ is a zero row or δ is a zero column. No zero row or column implies α is a zero column and γ is a zero row, or β is a zero row and δ is a zero column. So the 1s in each row and column occur consecutively. The second conclusion follows from the nonexistence of the submatrices of order 2.

Forbidden Submatrix Characterization

Theorem: A $(0,1)$-matrix is TNN if and only if it does not have any submatrix equal to one of

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The proof is by induction starting from a double staircase pattern.

Eigenvalue 0 of Irreducible Square $(0,1)$-TNN matrices

Since TNN matrices have all eigenvalues nonnegative, it follows from a previous observation that **if A is an irreducible $(0,1)$-TNN matrix of order $n \geq 2$, then 0 is an eigenvalue of A.**

Theorem: In fact, the **multiplicity of 0** as an eigenvalue is **at least**

$$\left\lfloor \frac{n}{2} \right\rfloor .$$

This follows from showing that a principal submatrix of order $k \geq \left\lceil \frac{n}{2} \right\rceil + 1$ of A has determinant equal to 0, and thus $x^{\lfloor n/2 \rfloor}$ is a factor of the characteristic polynomial.

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$(0,1)$-Hessenberg Matrices

These are square $(0,1)$-matrices of the form

$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$ (all 0s above superdiagonal), $* = 0$ or 1.

In a **full $(0,1)$-Hessenberg matrix** H_n of order n the $*$s all equal 1 ($* = 1$):

$$H_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

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Full $(0,1)$-Hessenberg Matrices

Theorem: The **characteristic polynomial of H_n** equals:

$$q_n(\lambda) = \sum_{k=0}^{\lceil \frac{n}{2} \rceil} (-1)^k \binom{n+1-k}{k} \lambda^{n-k}.$$

The $\pm$ coefficient of λ^{n-k} is the number of subsequences of $1, 2, \ldots, n$ of length k with no two numbers in the subsequence consecutive, and these sequences are in one-to-one correspondence with the principal submatrices of order k with a nonzero determinant (which is always 1).

Corollary: **0** is an **eigenvalue** of H_n with **multiplicity** $\lfloor \frac{n}{2} \rfloor$.

Remark: Since the coefficients of $q_n(\lambda)$ alternate in sign, H_n cannot have any negative roots—a property we knew since H_n is TNN, and thus cannot have any non-real roots as well.

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Characterization of Full $(0, 1)$-Hessenberg Matrices

Full $(0, 1)$-Hessenberg matrices are clearly irreducible.

Theorem: Let X_n be an irreducible, TNN $(0, 1)$-Hessenberg matrix of order n. Then $X_n = H_n$.

Thus H_n is the only irreducible $(0, 1)$ Hessenberg matrix of order n which is TNN.

Reason: Irreducibility with double staircase pattern implies that X_n has 1s everywhere on the superdiagonal, diagonal, and subdiagonal. The forbidden matrices of orders 2 and 3 imply that X_n is full.

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Extreme $(0,1)$-TNN Matrices

The full (0,1)-Hessenberg matrix H_n is a TNN with $(n-1)(n-2)/2$ 0s. This leads to the **Question: How many 0s can an irreducible $(0,1)$-TNN matrix of order n have?**

TNN and irreducibility imply double staircase pattern with 1s everywhere on superdiagonal, diagonal, subdiagonal. But this does not guarantee TNN: Recall the forbidden matrix

$$F_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Let z_n denote the **maximum number of 0s in an irreducible $(0,1)$-TNN matrix**. Note that the number of 1s in such a matrix is at least $(n-1)+n+(n-1)=3n-2$, and so the number of 0s is at most $n^2-(3n-2)=(n-1)(n-2)$.

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Extreme $(0,1)$-TNN Matrices, continued

Let X and Y be matrices of orders k and l, respectively, where the submatrix of order 2 of X in its lower right is J_2 (all 1s) and the submatrix of order 2 of Y in its upper left is also J_2. Then $X * Y$ is the matrix of order $k+l-2$ obtained by **'joining'** X and Y at their J_2s, and setting every other entry equal to 0.

Example: If $X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & \mathbf{1} & \mathbf{1} \\ 1 & \mathbf{1} & \mathbf{1} \end{bmatrix}$ and $Y = \begin{bmatrix} \mathbf{1} & \mathbf{1} & 1 \\ \mathbf{1} & \mathbf{1} & 1 \\ 0 & 1 & 1 \end{bmatrix}$, then

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Let X and Y be matrices of orders k and l, respectively, where the submatrix of order 2 of X in its lower right is J_2 (all 1s) and the submatrix of order 2 of Y in its upper left is also J_2. Then $X * Y$ is the matrix of order $k + l - 2$ obtained by **'joining'** X and Y at their J_2s, and setting every other entry equal to 0.

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$$X * Y = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & 1 \\ 1 & \mathbf{1} & \mathbf{1} & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Extreme $(0,1)$-TNN Matrices, continued

Let a matrix of order n be defined by: $A_n = A_3 * A_3^t * A_3 * A_3^t * \cdots$ (ends with A_3^t (n even) and with A_3 (n odd)) where e.g.

$$A_5 = A_3 * A_3^t * A_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & 1 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Theorem: Let A be an irreducible $(0,1)$-TNN matrix of order n with $z_n = (n-2)^2$ 0s. Then

$$A = A_n \text{ or } A_n^t.$$

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Spectrum of Extreme $(0,1)$-TNN Matrices

A **generalized full (0,1)-Hessenberg matrix** is a matrix of the form

$$H_{k_1} * H_{k_2}^t * H_{k_3} * H_{k_4}^t * \cdots \text{ or } H_{k_1}^t * H_{k_2} * H_{k_3}^t * H_{k_4} * \cdots$$

Lemma: All generalized full (0,1)-Hessenberg matrices of order n have the **same spectrum** and thus the same spectrum as H_n. In particular, the extremal matrices A_n have the same spectrum as H_n.

Note that the ranks of H_n and A_n are different: $\text{rank}(H_n) = n - 1$ but $\text{rank}(A_n) = \lceil \frac{n}{2} \rceil$.

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Spectrum of Extreme $(0,1)$-TNN Matrices,continued

Theorem: The **minimum Perron value** (spectral radius) of an irreducible, $(0,1)$-TNN matrix A of order n is $\mathbf{2+2\cos\frac{2\pi}{n+2}}$, achieved if and only if A is a **generalized full $(0,1)$-Hessenberg matrix**.

Corollary: The irreducible $(0,1)$-TNN matrices of order n with the minimum Perron value all have the same spectrum.

Spectrum of Extreme $(0,1)$-TNN Matrices,continued

Theorem: The **minimum Perron value** (spectral radius) of an irreducible, $(0,1)$-TNN matrix A of order n is $\mathbf{2+2\cos\frac{2\pi}{n+2}}$, achieved if and only if A is a **generalized full $(0,1)$-Hessenberg matrix**.

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Final Remark

Reference: RAB and S. Kirkland, *LAA*, 432 (2010), 1650–1662,

We have concentrated on TNN (0,1)-matrices as a way to investigate (0,1)-matrices with all eigenvalues nonnegative.

It's more difficult to get a handle on arbitrary (0,1)-matrices with nonnegative eigenvalues, but we are planning to continue our investigations.

Life is Good!