

On commuting matrices in max algebra and in classical nonnegative algebra

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Theorem

If $AB = BA$ then the eigenvalues α^j, β^j of A, B can be ordered so that for any polynomial $p(x, y)$ the eigenvalues of $p(A, B)$ are $p(\alpha^j, \beta^j)$, $j = 1, \dots, n$.

Frobenius 1878

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- **MX:** max (nonnegative) linear algebra

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Stephane Gaubert 1997:

The spectral theory in **MX** *"is extremely similar to the well-known Perron-Frobenius theory"* in **NN**

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- **NN**: (classical) nonnegative linear algebra
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Stephane Gaubert 1997:

The spectral theory in **MX** *"is extremely similar to the well-known Perron-Frobenius theory"* in **NN** with some important differences.

$$A \in \mathbb{R}_+^{n \times n} = A \geq 0$$

Definition

Eigenvalue α of A is a *distinguished* eigenvalue if there is an associated *nonnegative* eigenvector.

*eigenvalue = distinguished eigenvalue

*eigenvector = nonnegative eigenvector

A reducible:

$$P^{-1}AP = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

A_{11} , A_{22} square

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*Let $A \geq 0$ be irreducible. Then A has a unique *eigenvalue $\lambda(A)$ with an (ess) unique associated *eigenvector, which is positive.*

$\lambda(A)$ is the Perron root

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Theorem

*Let $A \geq 0$. Then A has *eigenvalue(s) (and *eigenvectors)*

Denote the largest *eigenvalue of A by $\lambda(A)$

$$a, b \geq 0$$

$$a \oplus b = \max(a, b)$$

$$a \otimes b = ab$$

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$$A, B \in \mathbb{R}_+^{n \times n}$$

$$C = A \oplus B: \quad c_{ij} = a_{ij} \oplus b_{ij}$$

$$C = A \otimes B: \quad c_{ij} = \bigoplus_k a_{ik} b_{kj}$$

$$a, b \geq 0$$

$$a + b = \max(a, b)$$

$$ab = ab$$

$$a, b \geq 0$$

$$a + b = \max(a, b)$$

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$$C = A + B: \quad c_{ij} = a_{ij} + b_{ij}$$

$$C = AB: \quad c_{ij} = \bigoplus_k a_{ik} b_{kj}$$

Definition

α is an eigenvalue of A :

$$\exists x \succeq 0, Ax = \alpha x$$

x is an eigenvector corr. α

Definition

α is an eigenvalue of A :

$$\exists x \neq 0, Ax = \alpha x$$

x is an eigenvector corr. α

Theorem

Let $A \geq 0$ be irreducible. Then A has a unique eigenvalue $\lambda(A)$ with associated eigenvectors, which are positive.

$\lambda(A)$ is the Perron root
max cycle geom mean

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{3}{4} & 1 \end{pmatrix}$$

$$(A^2)_{11} = \max\{1, 3/8\} = 1$$

$$(A^2)_{21} = \max\{3/4, 3/4\} = 3/4$$

$$A^2 = A$$

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Both columns are eigenvectors

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Theorem

Let $A \geq 0$. Then A has eigenvalue(s) (and eigenvectors)

$$\mathbf{CM}: \quad \begin{array}{cc} \text{matrix} & \text{evecs} \\ \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \end{array}$$

	matrix	evecs
CM :	$\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$
NN :	$\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & . & 0 \\ 1 & . & 0 \\ 1 & . & 1 \end{pmatrix}$

	matrix	evector
CM :	$\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$
NN :	$\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & . & 0 \\ 1 & . & 0 \\ 1 & . & 1 \end{pmatrix}$
MX :	$\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 4 & . & 0 \\ 3 & . & 0 \\ 1 & . & 1 \end{pmatrix}$

	matrix	evector
CM :	$\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$
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MX :	$\begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 4 & . & 0 \\ 3 & . & 0 \\ 1 & . & 1 \end{pmatrix}$

NN, MX : evaluates 4, 2

Theorem

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Theorem

If $AB = BA$ then A and B have a common eigenvector.

CM: Complex matrices – X basis of space for evaluate α

Proof.

$$AX = \alpha X$$

$$A(BX) = B(AX) = \alpha BX$$

$$BX = XC$$

$$Cz = \beta z, \quad z \neq 0$$

$$B(Xz) = X(Cz) = \beta Xz, \quad Xz \neq 0$$

$$A(Xz) = (AX)z = \alpha Xz \quad \square$$

Theorem

If $AB = BA$ then A and B have a common eigenvector.

NN: classic nonneg – X extremals of convex econe for
evalue α

Proof.

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$$A(BX) = B(AX) = \alpha BX$$

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Theorem

If $AB = BA$ then A and B have a common eigenvector.

MX: max nonneg – X extremals of max econe for eval
 α

Proof.

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Theorem

Let A_1, \dots, A_r be pairwise commuting matrices. Then for each eigenvalue α^i of A_i there exists an eigenvector x which is an eigenvector of all the A_j .

Theorem

If $AX = XB$ and

CM: the cols of X are lin indep

NN & MX: no col of X is 0

then every eval of B is an eval of A .

Proof.

$$Bz = \beta z, z \neq 0$$

$$AXz = XBz = \beta Xz, Xz \neq 0 \quad \square$$

Theorem

CM: Suppose that $A_1, \dots, A_r \in \mathbb{R}_+^{n \times n}$ pairwise commute. For $i = 1, \dots, r$, let the eigenvalues of A_i be α_i^j for $j = 1, \dots, n$. Let $p(x_1, \dots, x_r)$ be a polynomial. Then, the eigenvalues α_i^j can be ordered so that the eigenvalues of $p(A_1, \dots, A_r)$ are $p(\alpha_1^j, \dots, \alpha_r^j)$ for $j = 1, \dots, n$.
Frobenius 1896, Schur 1902

Theorem

MX: & NN Let $A_1, \dots, A_r \in \mathbb{R}_+^{n \times n}$ commute in pairs and let $p(x_1, \dots, x_r)$ be a polynomial

NN: such that $p(A_1, \dots, A_r) \geq 0$

Then,

- (i) For each $i \in \{1, \dots, r\}$ and evaluate α_i of A_i there exist values α_j of A_j for all $j \neq i$ such that $p(\alpha_1, \dots, \alpha_r)$ is an evaluate of $p(A_1, \dots, A_r)$;
- (ii) For each evaluate λ of $p(A_1, \dots, A_r)$ there exist values α_i of A_i for all $i = 1, \dots, r$ such that $\lambda = p(\alpha_1, \dots, \alpha_r)$.

$$A \leftarrow P^{-1}AP = \begin{pmatrix} A_{11} & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{(k-1)1} & A_{(k-1)2} & \cdots & A_{(k-1)(k-1)} & 0 \\ A_{k1} & A_{k2} & \cdots & A_{k(k-1)} & A_{kk} \end{pmatrix}$$

A_{ij} irreducible

Reduced graph $\mathcal{R}(A)$

$$V = \{1, \dots, k\}$$

$$i \rightarrow j \in E : A_{ij} \neq 0$$

Path from i to j

$$i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{p-1} \rightarrow i_m$$

Transitive closure $\mathcal{R}^*(A)$

$$i \xrightarrow{*} j : \text{exists path from } i \text{ to } j$$

Skeleton $\mathcal{S} = \mathcal{R}_*(A)$

$$(i, j) \in \mathcal{S} : i \xrightarrow{*} k \xrightarrow{*} j \text{ implies } k = i \text{ or } k = j$$

Example

$$\begin{pmatrix} \spadesuit & 0 & 0 & 0 \\ \clubsuit & \spadesuit & 0 & 0 \\ 0 & \clubsuit & \spadesuit & 0 \\ \clubsuit & \clubsuit & 0 & \spadesuit \end{pmatrix}$$



irred block

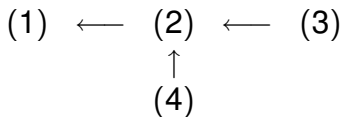


nonzero block

Example

$$\begin{pmatrix} \spadesuit & 0 & 0 & 0 \\ \clubsuit & \spadesuit & 0 & 0 \\ 0 & \clubsuit & \spadesuit & 0 \\ \clubsuit & \clubsuit & 0 & \spadesuit \end{pmatrix}$$

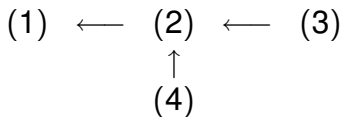
- \spadesuit irred block
 \clubsuit nonzero block



Example

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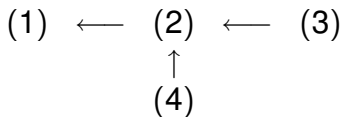
\spadesuit irred block
 \clubsuit nonzero block



$$\begin{pmatrix} \spadesuit & 0 & 0 & 0 \\ \clubsuit & \spadesuit & 0 & 0 \\ \heartsuit & \clubsuit & \spadesuit & 0 \\ \heartsuit & \clubsuit & 0 & \spadesuit \end{pmatrix}$$

\heartsuit in trans closure of skeleton

$$\begin{pmatrix} \spadesuit & 0 & 0 & 0 \\ \clubsuit & \spadesuit & 0 & 0 \\ \heartsuit & \clubsuit & \spadesuit & 0 \\ \heartsuit & \clubsuit & 0 & \spadesuit \end{pmatrix}$$



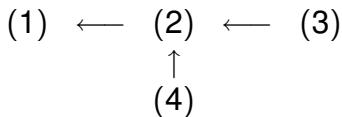
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$$\begin{array}{ccccc} (1) & \longleftarrow & (2) & \longleftarrow & (3) \\ & & \uparrow & & \\ & & (4) & & \end{array}$$

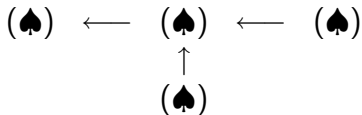
Identify:

node of $\mathcal{R}(A)$ - irred block of A

$$\begin{pmatrix} \spadesuit & 0 & 0 & 0 \\ \clubsuit & \spadesuit & 0 & 0 \\ \heartsuit & \clubsuit & \spadesuit & 0 \\ \heartsuit & \clubsuit & 0 & \spadesuit \end{pmatrix}$$



Identify:
 node of $\mathcal{R}(A)$ - irred block of A
 CLASS of A



Definition

A class A_{ij} of A is called *spectral* if $\lambda(A_{ij})$ is an eigenvalue of A and there is a evector x such that $x_i \neq 0$ if and only if $i \xrightarrow{*} j$ in $\mathcal{R}^*(A)$

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Theorem

Assume that A is in Frobenius form with Perron root α_j of A_{jj} . Then

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Theorem

Assume that A is in Frobenius form with Perron root α_j of A_{jj} . Then

NN: A_{ij} is spectral if and only if $i \xrightarrow{*} j$ in $\mathcal{R}^*(A)$ implies that $\alpha_i < \alpha_j$.

(Frobenius 1912, Victory 1985)

Definition

A class A_{ij} of A is called *spectral* if $\lambda(A_{ij})$ is an eigenvalue of A and there is a evector x such that $x_i \neq 0$ if and only if $i \xrightarrow{*} j$ in $\mathcal{R}^*(A)$

Theorem

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MX: A_{ij} is spectral if and only if $i \xrightarrow{*} j$ in $\mathcal{R}^*(A)$ implies that $\alpha_i \leq \alpha_j$.

Gaubert 1992, Butkovic (book) 2010

Theorem (NN:, MX:)

Assume that A is in Frobenius form with Perron roots α_k of A_{kk} pairwise distinct. Let A_{jj} be a spectral class of A .

Theorem (NN:, MX:)

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Theorem (NN:, MX:)

Assume that A is in Frobenius form with Perron roots α_k of A_{kk} pairwise distinct. Let A_{ij} be a spectral class of A . Then there exists a eigenvector x such that $x_i \neq 0$ if and only if $i \xrightarrow{*} j$.

MX:

$$\begin{pmatrix} 10^* & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 3 & 3^* \end{pmatrix} \quad \begin{pmatrix} 2 & . & 0 \\ 1 & . & 0 \\ 1 & . & 1 \end{pmatrix}$$

Skeleton *Spectral

$$(*1) \leftarrow (2) \leftarrow (*3)$$

Theorem

NN : & (MX: + is MAX !)

Suppose that $A_1, \dots, A_r \in \mathbb{R}_+^{n \times n}$ pairwise commute and that distinct classes of A_i , for $i = 1, \dots, r$, have distinct Perron roots. Then,

Theorem

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Suppose that $A_1, \dots, A_r \in \mathbb{R}_+^{n \times n}$ pairwise commute and that distinct classes of A_i , for $i = 1, \dots, r$, have distinct Perron roots. Then,

- 1. The classes of A_1, \dots, A_r and $A_1 + \dots + A_r$ coincide.*

Theorem

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Suppose that $A_1, \dots, A_r \in \mathbb{R}_+^{n \times n}$ pairwise commute and that distinct classes of A_i , for $i = 1, \dots, r$, have distinct Perron roots. Then,

- 1. The classes of A_1, \dots, A_r and $A_1 + \dots + A_r$ coincide.*
- 2. The transitive closures of the reduced graphs of all matrices A_1, \dots, A_r and $A_1 + \dots + A_r$ coincide.*

Theorem

3. *The spectral classes of the matrices A_1, \dots, A_r coincide*

MX: *and also coincide with the spectral classes of A_1, \dots, A_r and $A_1 + \dots + A_r$.*

In particular, A_1, \dots, A_r have the same number of distinct eigenvalues, which we denote by m .

Theorem

4. For $i = 1, \dots, r$, let the (distinct) eigenvalues of A_{ij} be α_i^j for $j = 1, \dots, m$.

MX: Let $p(x_1, \dots, x_r)$ be a non-constant max-polynomial.

NN: Let $p(x_1, \dots, x_r)$ be a non-constant polynomial such that $p(A_1, \dots, A_r) \geq 0$.

Then, the eigenvalues α_i^j can be ordered so that the eigenvalues of $p(A_1, \dots, A_r)$ are precisely $p(\alpha_1^j, \dots, \alpha_r^j)$ for $j = 1, \dots, m$.

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \quad \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$$

Assume in Frobenius form, B_{11} and B_{22} no common
evalue.

Compare $(AB)_{21}$ and $(BA)_{21}$

Assume $B_{21} = 0$

$$(AB)_{21} = A_{21}B_{11}$$

$$(BA)_{21} = B_{22}A_{21}$$

$$A_{21} = 0$$

$$A = \begin{pmatrix} 10 & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 3 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

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Skeleton *Spectral

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Skeleton *Spectral

$$(*1) \leftarrow (2) \leftarrow (*3)$$

$$AB = BA = \begin{pmatrix} 30 & 0 & 0 \\ 15 & 0 & 0 \\ 9 & 6 & 6 \end{pmatrix}$$

$$A = \begin{pmatrix} 10 & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 3 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Skeleton, *Spectral

$$(1^*) \leftarrow (2) \leftarrow (3^*)$$

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Skeleton, *Spectral

$$(1^*) \leftarrow (2) \leftarrow (3^*)$$

*Eigenvectors of A , B and AB .

$$\begin{pmatrix} 2 & . & 0 \\ 1 & . & 0 \\ 1 & . & 1 \end{pmatrix}$$

If you're interested, see

R. Katz, H. Schneider, S. Sergeev

On commuting matrices in max algebra
and in classical nonnegative algebra

<http://www.math.wisc.edu/hans/>

My Papers

Paper 160

THANKS

**THANKS
FOR THE HONOR YOU HAVE DONE ME**

**THANKS
FOR THE HONOR YOU HAVE DONE ME
and even more**

THANKS
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FOR LISTENING TO ME

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FOR LISTENING TO ME
If you listened