On commuting matrices in max algebra and in classical nonnegative algebra

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joint with Ricardo Katz and Sergei Sergeev
with input from Peter Butkovic

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Three classical theorems

Theorem

If $AB = BA$ then the eigenvalues α^j, β^j of A, B can be ordered so that for any polynomial $p(x, y)$ the eigenvalues of $p(A, B)$ are $p(\alpha^j, \beta^j)$, $j = 1, \ldots, n$.

Frobenius 1878

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- **CM:** complex linear algebra

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The spectral theory in **MX** *"is extremely similar to the well-known Perron-Frobenius theory"* in **NN**

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- **NN:** (classical) nonnegative linear algebra
- **MX:** max (nonnegative) linear algebra

Stephane Gaubert 1997:
The spectral theory in **MX** *"is extremely similar to the well-known Perron-Frobenius theory"* in **NN**
with some important differences.

$$A \in \mathbb{R}_+^{n\times n} \quad = \quad A \geq 0$$

Definition

Eigenvalue α of A is a *distinguished* eigenvalue if there is an associated *nonnegative* eigenvector.

*eigenvalue = distinguished eigenvalue
*eigenvector = nonnegative eigenvector

NN: Classic PERRON-FROBENIUS 1912

A reducible:

$$P^{-1}AP = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

$A_{11}, \quad A_{22} \quad$ square

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Theorem

*Let $A \geq 0$. Then A has *eigenvalue(s) (and *eigenvectors)*

Denote the largest *eigenvalue of A by $\lambda(A)$

$$a, b \geq 0$$

$$a \oplus b = \max(a, b)$$

$$a \otimes b = ab$$

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$$A, B \in \mathbb{R}_+^{n \times n}$$

$$C = A \oplus B: \quad c_{ij} = a_{ij} \oplus b_{ij}$$

$$C = A \otimes B: \quad c_{ij} = \bigoplus_k a_{ik} b_{kj}$$

$$a, b \geq 0$$

$$a + b = \max(a, b)$$

$$ab = ab$$

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$\lambda(A)$ is the Perron root
max cycle geom mean

Warning

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{3}{4} & 1 \end{pmatrix}$$

$$(A^2)_{11} = \max\{1, 3/8\} = 1$$

$$(A^2)_{21} = \max\{3/4, 3/4\} = 3/4$$

$$A^2 = A$$

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Let $A \geq 0$. Then A has eigenvalue(s) (and eigenvectors)

$$\begin{array}{lcc} & \text{matrix} & \text{evectors} \\ \textbf{CM}: & \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \end{array}$$

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NN, MX : evalues 4, 2

Theorem

If $AB = BA$ then A and B have a common eigenvector.

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If $AB = BA$ then A and B have a common eigenvector.

CM: Complex matrices – X basis of espace for evalue α

Proof.

$$AX = \alpha X$$

$$A(BX) = B(AX) = \alpha BX$$

$$BX = XC$$

$$Cz = \beta z, \quad z \neq 0$$

$$B(Xz) = X(Cz) = \beta Xz, \quad Xz \neq 0$$

$$A(Xz) = (AX)z = \alpha Xz \quad \square$$

Theorem

If $AB = BA$ then A and B have a common eigenvector.

NN: classic nonneg – X extremals of convex econe for evalue α

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Theorem

If $AB = BA$ then A and B have a common eigenvector.

MX: max nonneg – X extremals of max econe for evalue α

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Theorem

Let $A_1, \ldots, A_r$ be pairwise commuting matrices. Then for each eigenvalue α^i of A_i there exists an eigenvector x which is an eigenvector of all the A_j.

Theorem

If $AX = XB$ and

CM: *the cols of X are lin indep*

NN *&* **MX:** *no col of X is* 0

then every evalue of B is an evalue of A.

Proof.

$$Bz = \beta z, \; z \neq 0$$

$$AXz = XBz = \beta Xz, \; Xz \neq 0 \qquad \square$$

Theorem

CM: *Suppose that* $A_1, \dots, A_r \in \mathbb{R}_+^{n \times n}$ *pairwise commute. For* $i = 1, \dots, r$, *let the eigenvalues of* A_i *be* α_i^j *for* $j = 1, \dots, n$. *Let* $p(x_1, \dots, x_r)$ *be a polynomial. Then, the eigenvalues* α_i^j *can be ordered so that the eigenvalues of* $p(A_1, \dots, A_r)$ *are* $p(\alpha_1^j, \dots, \alpha_r^j)$ *for* $j = 1, \dots, n$.
Frobenius 1896, Schur 1902

Theorem

MX: & NN *Let* $A_1, \ldots, A_r \in \mathbb{R}_+^{n\times n}$ *commute in pairs and let* $p(x_1, \ldots, x_r)$ *be a polynomial*
NN: *such that* $p(A_1, \ldots, A_r) \geq 0$
Then,

(i) *For each* $i \in \{1, \ldots, r\}$ *and evalue* α_i *of* A_i *there exist evalues* α_j *of* A_j *for all* $j \neq i$ *such that* $p(\alpha_1, \ldots, \alpha_r)$ *is an evalue of* $p(A_1, \ldots, A_r)$*;*

(ii) *For each evalue* λ *of* $p(A_1, \ldots, A_r)$ *there exist evalues* α_i *of* A_i *for all* $i = 1, \ldots, r$ *such that* $\lambda = p(\alpha_1, \ldots, \alpha_r)$*.*

Frobenius form 1912

$$A \leftarrow P^{-1}AP = \begin{pmatrix} A_{11} & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{(k-1)1} & A_{(k-1)2} & \cdots & A_{(k-1)(k-1)} & 0 \\ A_{k1} & A_{k2} & \cdots & A_{k(k-1)} & A_{kk} \end{pmatrix}$$

A_{ii} irreducible

Reduced graph $\mathcal{R}(A)$

$$V = \{1,, \ldots, k\}$$

$$i \to j \in E : \; A_{ij} \neq 0$$

Path from i to j

$$i_0 \to i_1 \to \cdots \to i_{p-1} \to i_m$$

Transitive closure $\mathcal{R}^*(A)$

$$i \xrightarrow{*} j : \; \text{exists path from } i \text{ to } j$$

Skeleton $S = \mathcal{R}_*(A)$

$$(i, j) \in S : i \xrightarrow{*} k \xrightarrow{*} j \text{ implies } k = i \text{ or } k = j$$

Example

$$\begin{pmatrix} \spadesuit & 0 & 0 & 0 \\ \clubsuit & \spadesuit & 0 & 0 \\ 0 & \clubsuit & \spadesuit & 0 \\ \clubsuit & \clubsuit & 0 & \spadesuit \end{pmatrix}$$

$\spadesuit$ irred block
$\clubsuit$ nonzero block

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$$\begin{pmatrix} \spadesuit & 0 & 0 & 0 \\ \clubsuit & \spadesuit & 0 & 0 \\ \heartsuit & \clubsuit & \spadesuit & 0 \\ \heartsuit & \clubsuit & 0 & \spadesuit \end{pmatrix}$$

$\heartsuit$ in trans closure of skeleton

Identification

$$\begin{pmatrix} \spadesuit & 0 & 0 & 0 \\ \clubsuit & \spadesuit & 0 & 0 \\ \color{red}{\heartsuit} & \clubsuit & \spadesuit & 0 \\ \color{red}{\heartsuit} & \clubsuit & 0 & \spadesuit \end{pmatrix}$$

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Identify:

node of $\mathcal{R}(A)$ - irred block of A

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CLASS of A

$$\begin{array}{ccccc} (\spadesuit) & \longleftarrow & (\spadesuit) & \longleftarrow & (\spadesuit) \\ & & \uparrow & & \\ & & (\spadesuit) & & \end{array}$$

Definition

A class A_{ii} of A is called *spectral* if $\lambda(A_{ii})$ is an eigenvalue of A and there is a evector x such that $x_i \neq 0$ if and only if $i \xrightarrow{*} j$ in $\mathcal{R}^*(A)$

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Assume that A is in Frobenius form with Perron root α_j of A_{jj}. Then

NN: *A_{jj} is spectral if and only if $i \xrightarrow{*} j$ in $\mathcal{R}^*(A)$ implies that $\alpha_i < \alpha_j$.*
(Frobenius 1912, Victory 1985)

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MX: *A_{jj} is spectral if and only if $i \xrightarrow{*} j$ in $\mathcal{R}^*(A)$ implies that $\alpha_i \leq \alpha_j$.*
Gaubert 1992, Butkovic (book) 2010

Theorem (NN:, MX:)

Assume that A is in Frobenius form with Perron roots α_k of A_{kk} pairwise distinct. Let A_{jj} be a spectral class of A.

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Assume that A is in Frobenius form with Perron roots α_k of A_{kk} pairwise distinct. Let A_{jj} be a spectral class of A. Then there exists a eigenvector x such that $x_i \neq 0$ if and only if $i \xrightarrow{*} j$.

MX:

$$\begin{pmatrix} 10^* & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 3 & 3^* \end{pmatrix} \quad \begin{pmatrix} 2 & . & 0 \\ 1 & . & 0 \\ 1 & . & 1 \end{pmatrix}$$

Skeleton $\quad$ *Spectral

$$(^*1) \leftarrow (2) \leftarrow (^*3)$$

Theorem

NN : & (MX: + is MAX !)

Suppose that $A_1, \ldots, A_r \in \mathbb{R}_+^{n\times n}$ *pairwise commute and that distinct classes of* A_i*, for* $i = 1, \ldots, r$*, have distinct Perron roots. Then,*

Theorem

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Suppose that $A_1, \ldots, A_r \in \mathbb{R}_+^{n \times n}$ pairwise commute and that distinct classes of A_i, for $i = 1, \ldots, r$, have distinct Perron roots. Then,

1. *The classes of $A_1, \ldots, A_r$ and $A_1 + \ldots + A_r$ coincide.*

Theorem

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Suppose that $A_1, \ldots, A_r \in \mathbb{R}_+^{n\times n}$ pairwise commute and that distinct classes of A_i, for $i = 1, \ldots, r$, have distinct Perron roots. Then,

1. *The classes of $A_1, \ldots, A_r$ and $A_1 + \ldots + A_r$ coincide.*
2. *The transitive closures of the reduced graphs of all matrices $A_1, \ldots, A_r$ and $A_1 + \ldots + A_r$ coincide.*

Theorem

3. *The spectral classes of the matrices* $A_1, \ldots, A_r$ *coincide*
   MX: *and also coincide with the spectral classes of* $A_1, \ldots, A_r$ *and* $A_1 + \ldots + A_r$.
   In particular, $A_1, \ldots, A_r$ *have the same number of distinct eigenvalues, which we denote by* m.

Theorem

4. *For $i = 1, \ldots, r$, let the (distinct) eigenvalues of A_{ii} be α_i^j for $j = 1, \ldots, m$.*
   MX: *Let $p(x_1, \ldots, x_r)$ be a non-constant max-polynomial.*
   NN: *Let $p(x_1, \ldots, x_r)$ be a non-constant polynomial such that $p(A_1, \ldots, A_r) \geq 0$.*
   Then, the eigenvalues α_i^j can be ordered so that the eigenvalues of $p(A_1, \ldots, A_r)$ are precisely $p(\alpha_1^j, \ldots, \alpha_r^j)$ for $j = 1, \ldots, m$.

ingredient of proof – reduced graphs

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \quad \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$$

Assume in Frobenius form, B_{11} and B_{22} no common evalue.

Compare $(AB)_{21}$ and $(BA)_{21}$

Assume $B_{21} = 0$

$$(AB)_{21} = A_{21}B_{11}$$

$$(BA)_{21} = B_{22}A_{21}$$

$$A_{21} = 0$$

NN: example

$$A = \begin{pmatrix} 10 & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 3 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

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$$AB = BA = \begin{pmatrix} 30 & 0 & 0 \\ 15 & 0 & 0 \\ 9 & 6 & 6 \end{pmatrix}$$

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Skeleton, *Spectral

$$(1^*) \leftarrow (2) \leftarrow (3^*)$$

*Eigenvectors of A, B and AB.

$$\begin{pmatrix} 2 & . & 0 \\ 1 & . & 0 \\ 1 & . & 1 \end{pmatrix}$$

If you're interested, see

R. Katz, H. Schneider, S. Sergeev

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and in classical nonnegative algebra

http://www.math.wisc.edu/ hans/
My Papers
Paper 160

THANKS

THANKS
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If you listened